I. NEWTONIAN MECHANICS

Newton's laws of motion

1. There exists a primary inertial frame of reference. In this reference frame every body remains at rest, or in uniform motion, unless acted upon by a force, $\boldsymbol{F}$. The condition $\boldsymbol{F} = \boldsymbol{0}$ thus implies a constant velocity, $\boldsymbol{v}$, and a constant momentum, $\boldsymbol{p} = m\boldsymbol{v}$.
2. In the primary inertial frame, application of a force alters the momentum. This alteration is given quantitatively as
$$\boldsymbol{F} = \frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} \equiv \dot{\boldsymbol{p}}$$
3. If body 1 exerts a force $\boldsymbol{F}_{12}$ on body 2, then body 2 exerts an equal and opposite force $\boldsymbol{F}_{21} = -\boldsymbol{F}_{12}$ on body 1.

Central forces

Conservative forces can be assigned a (not unique) scalar potential:

$$\boldsymbol{F} = -\nabla V$$

in the case of central forces, the potential is only dependent on the distance from the origin

$$\boldsymbol{F} = -\frac{\mathrm{d}V(r)}{\mathrm{d}r}\hat{\boldsymbol{r}}$$

In this case the three dimensional motion occurs in a plane, whose normal vector is the angular momentum, $\boldsymbol{\ell}$:

$$\boldsymbol{\ell} \equiv \boldsymbol{r} \times \boldsymbol{p}$$

i.e the plane of motion is spanned by the two vectors $\boldsymbol{p}$ and $\boldsymbol{r}$. The angular momentum for a central potential is a conserved quantity, with magnitude

$$\ell = mr^2\dot{\phi}$$

This leads to Kepler's second law, as this implies that the area of the ellipse that is swept per (small) interval of time is constant, thus that

$$\dot{A} = \frac{\ell}{2m}$$

The total energy of the system is given by

$$E = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\phi}^2\right) + V(r)$$

using the constancy of the angular momentum, this can be interpreted slightly differently, namely that

$$E = \frac{1}{2}m\dot{r}^2 + V_{eff}(r)$$

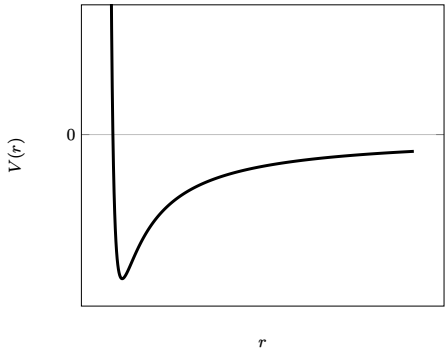
where

$$V_{eff}(r) = V(r) + \frac{\ell^2}{2mr^2}$$

The typical case is that

$$V(r) = -\frac{m\gamma}{r}$$

in which case we can plot the effective potential:



Figur 1

To solve for the motion analytically we use a change in variables, where we express the radius as a function of the angle, ϕ. In this case the energy is given as

$$E = \frac{\ell^2}{2mr^4}\left(\frac{\mathrm{d}r}{\mathrm{d}\phi}\right)^2 + \frac{\ell^2}{2mr^2} - \frac{m\gamma}{r}$$

Separation of variables yields

$$\phi = \phi_0 \mp \int^u \frac{\mathrm{d}u'}{\sqrt{\frac{2mE}{\ell^2} + \frac{2m^2\gamma u'}{\ell^2} - u'^2}}$$

Which can be solved using a substitution

$$\phi = \phi_0 \pm \arccos\left(\frac{1 - u\ell^2/m^2\gamma}{\sqrt{1 + 2E\ell^2/m^3\gamma^2}}\right)$$

Or, alternatively

$$r^{-1} = C\left(1 - \varepsilon\cos(\phi - \phi_0)\right)$$

where

$$\varepsilon = \sqrt{1 + \frac{2E\ell^2}{m^3\gamma^2}}, \quad C = \frac{m^2\gamma}{\ell^2}$$

ε is the eccentricity of the orbit. Here is the classification of orbits, as per eccentricity:

1. $\varepsilon = 0 \implies$ circular
2. $\varepsilon \in (0,1) \implies$ elliptic
3. $\varepsilon = 1 \implies$ parabolic
4. $\varepsilon > 1 \implies$ hyperbolic

II. ACCELERATED COORDINATE SYSTEMS

Rotating coordinate systems

$$\left(\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\right)_{\text{inertial}} = \left(\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\right)_{\text{body}} + \boldsymbol{\omega} \times \boldsymbol{r}$$

where the subscript *inertial* refers to a measurement of the velocity taken in an inertial reference frame, whereas *body* refers to the velocity of the body, taken in a reference frame that is rotating together with the body.

Accelerations

Similarly for rotating coordinate systems that do not have a constant angular velocity

$$\begin{aligned}\left(\frac{\mathrm{d}^2\boldsymbol{r}}{\mathrm{d}t^2}\right)_{\text{inertial}} = \left(\frac{\mathrm{d}^2\boldsymbol{r}}{\mathrm{d}t^2}\right)_{\text{body}} &+ 2\boldsymbol{\omega} \times \left(\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\right)_{\text{body}} \\ &+ \left(\frac{\mathrm{d}\boldsymbol{\omega}}{\mathrm{d}t}\right) \times \boldsymbol{r} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r})\end{aligned}$$

The corresponding fictitious forces in the rotating frame have been given the following names

$$\begin{aligned} -m\frac{\mathrm{d}\boldsymbol{\omega}}{\mathrm{d}t} \times \boldsymbol{r} &\quad \text{Euler force} \\ -2m\boldsymbol{\omega} \times \left(\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\right)_{\text{body}} &\quad \text{Coriolis force} \\ -m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}) &\quad \text{Centrifugal force}\end{aligned}$$

Translations

Suppose this arbitrarily rotating coordinate system additionally experiences a translational acceleration. Let the vector from the inertial reference frame's origin to the rotating origin be $\boldsymbol{a}$, then

$$\begin{aligned}\left(\frac{\mathrm{d}^2\boldsymbol{r}}{\mathrm{d}t^2}\right)_{\text{inertial}} = \left(\frac{\mathrm{d}^2\boldsymbol{r}}{\mathrm{d}t^2}\right)_{\text{body}} &+ 2\boldsymbol{\omega} \times \left(\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\right)_{\text{body}} \\ &+ \left(\frac{\mathrm{d}\boldsymbol{\omega}}{\mathrm{d}t}\right) \times \boldsymbol{r} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}) \\ &+ \left(\frac{\mathrm{d}^2\boldsymbol{a}}{\mathrm{d}t^2}\right)_{\text{inertial}}\end{aligned}$$

That is, the additional translational acceleration is merely an additive term, independent of rotation.

III. LAGRANGIAN DYNAMICS

One of the many benefits of Lagrangian dynamics is that it readily includes any constraints on the motion in the equations of motion, where Newtonian mechanics would have required careful attention to the additional forces and angles.

Holonomic constraints

Consider the motion of N particles, this system has $n = 3N$ degrees of freedom, as each particle is described with three coordinates. Now let us describe the coordinates of the particles as follows: The first particle's coordinates are given as x_1, x_2 and x_3, and then the m-th particle's coordinates are $x_{3(m-1)+1}$, $x_{3(m-1)+2}$ and $x_{3(m-1)+3}$. We can write k equations*, in the form

$$f_j(x_1, \dots, x_n, t) = c_j, \quad j = 1, 2, ..., k$$

that describe the constraints fully, then we call the constraints *holonomic*.

This leads to a natural change in coordinates, as the old coordinates $\{x_i\}_{i\in\{1,2,\dots,n\}}$ no longer are linearly independent, due to the constraints. Therefore there are $n - k$ *independent* degrees of freedom, which are referred to as **generalised coordinates**.

D'Alembert's principle is based on the observation that reaction forces — that is forces that are the result of constraints — cannot do any virtual work. This is due to the fact that $\delta W = \boldsymbol{F} \cdot \delta\boldsymbol{x}$, but the constraints state that there can be no virtual displacement parallel to constraining forces. The book refers to $\delta\boldsymbol{x}$'s that cannot be nonzero due to the constraining forces as *virtual displacements*. Imagine a ball on the surface of the table; the $\delta\boldsymbol{x}$ that goes into the table is a virtual displacement, because this displacement cannot happen due to the constraining forces. This allows us to state D'Alembert's principle:

$$\sum_i \left(F_i^{(a)} - \dot{p}_i\right) \delta x_i = 0$$

where $F_i^{(a)}$ is the i-th *applied* force.

NB: If there were no constraints then the δx_is would be linearly independent, and by definition of linear independence this would imply that $F_i = \dot{p}_i$†, which is exactly Newton's second law.

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* We are assuming that these k holonomic constraints are independent of each other. That is that, for example, there aren't two equations that describe the same constraints.

† We'll leave out the superscript (a) from now on, just like the book does.

However, due to the fact that we are dealing with constraining forces the δx_is are *not* linearly independent, which in turn means that we cannot conclude that each coefficient of δx_i is zero. D'Alembert's principle can be used to state the equations of motion, without mentioning the constraining forces at all:

The infinitesimal work done is given by

$$\delta W = \sum_i F_i \delta x_i$$

however, the δx_is are not linearly independent, due to the constraints, therefore it makes sense to express this in a linearly independent basis

$$\delta W = \sum_{\sigma=1}^{n-k} \left(\sum_{i=1}^{n} F_i \frac{\partial x_i}{\partial q_\sigma} \right) \delta q_\sigma$$

The coefficients of each δq_σ in the basis are referred to as the *generalised force*:

$$Q_\sigma \equiv \sum_{i=1}^{n} F_i \frac{\partial x_i}{\partial q_\sigma}$$

It only remains to rewrite the $\dot{p}_i$ term from Equation III. Once this is done, we get **Lagrange's equations**:

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial T}{\partial \dot{q}_\sigma} - \frac{\partial T}{\partial q_\sigma} = Q_\sigma, \quad \sigma = 1, 2, ..., n-k$$

This equation becomes particularly beautiful, if the generalised forces are conservative, as they then can be written as the gradient of a scalar potential, in which case we have the **Euler-Lagrange** equations:

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} = 0, \quad \sigma = 1, ..., n-k$$

where the **Lagrangian** is defined as

$$L \equiv T - V$$

the kinetic energy minus the potential energy.

Calculus of Variations

The Euler-Lagrange equations can be found through a more mathematical approach. Suppose we have a functional, $\phi^L(\boldsymbol{\gamma}, \dot{\boldsymbol{\gamma}})$, which is a function of the trajectory, and velocity. This functional, the *action*, is defined as

$$S \equiv \int_{t_1}^{t_2} L(\boldsymbol{q}, \dot{\boldsymbol{q}}, t) \, \mathrm{d}t$$

Hamilton's principle now states that this functional is stationary, if the trajectory $\boldsymbol{q}(t)$[‡] is a solution to Newton's laws of motion. Additionally we require that there is no variation of the position at t_1 and t_2, as we would like to decide where the particles start and finish:

$$\delta \boldsymbol{q}(t_1) = 0 = \delta \boldsymbol{q}(t_2)$$

The stationariness of the functional implies that

$$\delta S = \delta \int_{t_1}^{t_2} L(\boldsymbol{q}, \dot{\boldsymbol{q}}, t) \, \mathrm{d}t = 0$$

The variation of the action can be rewritten as

$$0 = \delta S = \int_{t_1}^{t_2} \sum_{\sigma=1}^{n} \left(\frac{\partial L}{\partial q_\sigma} \delta q_\sigma + \frac{\partial L}{\partial \dot{q}_\sigma} \delta \dot{q}_\sigma \right) \mathrm{d}t$$

And using integration by parts once on the second term, and applying our constraint that there is no variation at the endpoints of the trajectory, we yield

$$0 = \int_{t_1}^{t_2} \left(\sum_{\sigma} \left(\frac{\partial L}{\partial q_\sigma} - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \dot{q}_\sigma} \right) \delta q_\sigma \right) \mathrm{d}t$$

Now there are two possibilities

1. All q_σs are linearly independent of each other, in which case we get the old Euler-Lagrange equations

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} = 0$$

2. There are k holonomic constraints, f_j for $j = 1, 2..., k$, in which case we get

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} = \sum_{j=1}^{k} \lambda_j \frac{\partial f_j}{\partial q_\sigma}, \quad \sigma = 1, ..., n$$

This might look familiar: In MatIntro we learnt about finding extrema of functions at the border of its domain. Let f be the function whose extremum we want to find, and g be the function that describes the border of the domain, then

$$\nabla f = \lambda \nabla g$$

at the extremum! This technique is called the **method of Lagrange multipliers**.

Generalised Momenta

For a conservative, holonomic system, Lagrange's equations state that

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} = 0$$

[‡] Here the vector $\boldsymbol{q}(t)$ is a long vector which can be thought of as an ordered list of coordinates of all the particles.

for $n-k$ generalised coordinates, q_σ. We define the generalised momentum as

$$p_\sigma \equiv \frac{\partial L}{\partial \dot{q}_\sigma}$$

which in turn implies that

$$\dot{p}_\sigma = \frac{\partial L}{\partial q_\sigma}$$

Thus if $\frac{\partial L}{\partial q_\sigma} = 0$, p_σ is a conserved quantity. In cases where $\frac{\partial L}{\partial q_\sigma} = 0$ for some σ, we refer to q_σ as a *cyclic coordinate*.

The Lagrangian is invariant under point transformations (assuming they aren't too ugly). Suppose that our old coordinates, $\boldsymbol{q}$ and $\dot{\boldsymbol{q}}$, are given by the following transformations

$$\begin{aligned}\boldsymbol{q} &= \boldsymbol{q}(\boldsymbol{Q}, t)\\ \dot{\boldsymbol{q}} &= \dot{\boldsymbol{q}}(\boldsymbol{Q}, \dot{\boldsymbol{Q}}, t)\end{aligned}$$

In that case we can put this into the Lagrangian:

$$L(\boldsymbol{q}, \dot{\boldsymbol{q}}, t) = L(\boldsymbol{q}(\boldsymbol{Q}, t), \dot{\boldsymbol{q}}(\boldsymbol{Q}, \dot{\boldsymbol{Q}}, t), t) = \tilde{L}(\boldsymbol{Q}, \dot{\boldsymbol{Q}}, t)$$

Here we can again use the variational principle and get the Euler-Lagrange equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \tilde{L}}{\partial \dot{Q}_\sigma} - \frac{\partial \tilde{L}}{\partial Q_\sigma} = 0$$

VI. HAMILTONIAN DYNAMICS

The Hamiltonian is defined as the Legendre transformation of the Lagrangian:

$$H(\boldsymbol{q}, \boldsymbol{p}, t) \equiv \sum_\sigma p_\sigma \dot{q}_\sigma - L(\boldsymbol{q}, \dot{\boldsymbol{q}}, t)$$

The Legendre transformation is a transformation that exchanges one variable for another, in this case the transformation replaces $\dot{\boldsymbol{q}}$ with $\boldsymbol{p}$. Therefore, even though the Lagrangian is a function of $\boldsymbol{q}$ and $\dot{\boldsymbol{q}}$, the Hamiltonian is a function of $\boldsymbol{q}$ and $\boldsymbol{p}$.

Properties of the Hamiltonian:

1. If the Lagrangian does not depend explicitly on time, then the Hamiltonian is independent of time:

$$\frac{\partial L}{\partial t} = 0 \implies \frac{\mathrm{d}H}{\mathrm{d}t} = 0$$

2. If the potential and constraints are time-independent, then the Hamiltonian is the total energy of the system:

$$H = T + V$$

which is a conserved quantity.

An infinitesimal change in the Hamiltonian is given by

$$\mathrm{d}H = \sum_\sigma \left(\dot{q}_\sigma \mathrm{d}p_\sigma - \frac{\partial L}{\partial q_\sigma}\mathrm{d}q_\sigma \right) - \frac{\partial L}{\partial t}\mathrm{d}t$$

This leads to Hamilton's equations:

$$\frac{\partial H}{\partial p_\sigma} = \dot{q}_\sigma, \quad \frac{\partial H}{\partial q_\sigma} = -\dot{p}_\sigma, \quad \frac{\mathrm{d}H}{\mathrm{d}t} = \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}$$

Canonical Transformations

We know how to perform transformations using the Lagrange formalism, therefore let us begin with that. But before we can do that, we need to quickly discuss the Legendre transformation again. We know that

$$H = \sum_\sigma p_\sigma \dot{q}_\sigma - L$$

If we assume that L is convex in the variable $\dot{q}_\sigma$ (which is the case for most physical problems), then applying the Legendre transformation again will yield the original function, that is to say that[§]

$$L = \sum_\sigma p_\sigma \dot{q}_\sigma - H, \quad \dot{q}_\sigma = \frac{\partial H}{\partial p_\sigma}$$

This leads to

$$\begin{aligned}\sum_\sigma p_\sigma \dot{q}_\sigma - H(\boldsymbol{q}, \boldsymbol{p}, t) = \sum_\sigma P_\sigma \dot{Q}_\sigma - \tilde{H}(\boldsymbol{Q}, \boldsymbol{P}, t)\\ + \frac{\mathrm{d}}{\mathrm{d}t}F(\boldsymbol{q}, \boldsymbol{Q}, t)\end{aligned}$$

The assumption that $F = 0$ is too strict a constraint. It turns out that $F(\boldsymbol{q}, \boldsymbol{Q}, t)$ is a generator of the transformation $\boldsymbol{q} \rightsquigarrow \boldsymbol{Q}$, $\boldsymbol{p} \rightsquigarrow \boldsymbol{P}$, and that

$$\begin{aligned}p_\sigma &= \frac{\partial}{\partial q_\sigma}F(\boldsymbol{q}, \boldsymbol{Q}, t)\\ -P_\sigma &= \frac{\partial}{\partial Q_\sigma}F(\boldsymbol{q}, \boldsymbol{Q}, t)\\ \tilde{H}(\boldsymbol{Q}, \boldsymbol{P}, t) &= H(\boldsymbol{q}, \boldsymbol{p}, t) + \frac{\partial}{\partial t}F(\boldsymbol{q}, \boldsymbol{Q}, t)\end{aligned}$$

[§] This is not as trivial as it seems, as the full definition of a Legendre transformation is $f^*(p) = \sup_x \{xp - f(x)\}$, where it is *not* always the case that $f(x) = f^{**}(x) = \sup_p \{px - f^*(p)\}$.

Hamilton-Jacobi Theory

If we could choose a F such that Q_σ is cyclic for all σ, then this would make the description of the dynamics of the system trivial. It turns out if we choose

$$F(\boldsymbol{q}, \boldsymbol{Q}, t) = -\sum_\sigma P_\sigma Q_\sigma + S(\boldsymbol{q}, \boldsymbol{P}, t)$$

where S is the *action*, then this requirement is met. The action thus also generates a transformation:

$$p_\sigma = \frac{\partial}{\partial q_\sigma} S(\boldsymbol{q}, \boldsymbol{P}, t)$$
$$Q_\sigma = \frac{\partial}{\partial P_\sigma} S(\boldsymbol{q}, \boldsymbol{P}, t)$$

And

$$\frac{\partial}{\partial t} S(\boldsymbol{q}, \boldsymbol{P}, t) = \frac{\partial}{\partial t} F(\boldsymbol{q}, \boldsymbol{Q}, t)$$

which in turn implies that

$$\tilde{H}(\boldsymbol{Q}, \boldsymbol{P}, t) = H(\boldsymbol{q}, \boldsymbol{p}, t) + \frac{\partial}{\partial t} S(\boldsymbol{q}, \boldsymbol{P}, t) = 0$$

However, we cannot find the action until we have determined the trajectories. If we assume that

$$p_\sigma = \frac{\partial}{\partial q_\sigma} S(\boldsymbol{q}, \boldsymbol{P}, t)$$

then we can find a differential equation in S[¶]:

$$H(\boldsymbol{q}, \nabla_{\boldsymbol{q}} S, t) + \frac{\partial}{\partial t} S(\boldsymbol{q}, \boldsymbol{P}, t) = 0$$

The solution to this differential equation will have $n+1$ constants of integration (n from the integration of the gradient, and one from the temporal derivative). Solving this differential equation will give us S. We choose that[**]

$$\boldsymbol{P} = \boldsymbol{\alpha}$$

where $\alpha_1, \alpha_2, ..., \alpha_n$ are any n independent, non-additive integration constants. Additionally

$$\boldsymbol{Q} = \boldsymbol{\beta}$$

where $\beta_1, \beta_2, ..., \beta_n$ are n additional independent constants. Thus solving Equations VI and VI for $\boldsymbol{q}$ and $\boldsymbol{p}$ we get that

$$q_\sigma = q_\sigma(\alpha_1, ..., \alpha_n; \beta_1, ..., \beta_n; t), \quad \sigma = 1, ..., n$$

which is the trajectory of the particles.

Specifically, assume that the Hamiltonian has no explicit time-dependence, then the action can be split into the sum of two parts

$$S(\boldsymbol{q}, \boldsymbol{\alpha}, t) = W(\boldsymbol{q}, \boldsymbol{\alpha}) - \alpha_1 t$$

This gives us

$$H(\boldsymbol{q}, \nabla_{\boldsymbol{q}} W) = \alpha_1 = E$$

Action-Angle Variables

Suppose the action is separable, which implies that W can be written as:

$$W(\boldsymbol{q}, \boldsymbol{\alpha}) = W_1(q_1, \boldsymbol{\alpha}) + W_2(q_2, \boldsymbol{\alpha}) + \cdots + W_n(q_n, \boldsymbol{\alpha})$$

The set of problems to which this applies can be readily solved using Hamilton-Jacobi theory. This motion is additionally called *periodic* motion, if one of the two conditions is satisfied

1. (p, q) periodically returns to the same values.
2. p is a periodic function of q.

Let us now define the **action variables**:

$$J_\sigma \equiv \oint p_\sigma \, \mathrm{d}q_\sigma$$

The canonical momentum from Hamilton-Jacobi theory is

$$p_\sigma = \frac{\partial W}{\partial q_\sigma} = \frac{\partial W_\sigma}{\partial q_\sigma}, \quad \sigma = 1, ..., n$$

substituting this into Equation VI we get that

$$J_\sigma = J_\sigma(\boldsymbol{\alpha})$$

Thus J_σ is also a constant of motion. We can invert this

$$\alpha_\sigma = \alpha_\sigma(\boldsymbol{J}) \implies \alpha_1(\boldsymbol{J}) = E = H(\boldsymbol{J})$$

Let us now express W and S as functions of $\boldsymbol{J}$:

$$W(\boldsymbol{q}, \alpha_1(\boldsymbol{J}), \alpha_2(\boldsymbol{J}), ..., \alpha_n(\boldsymbol{J})) = \bar{W}(\boldsymbol{q}, \boldsymbol{J})$$
$$S(\boldsymbol{q}, \alpha_1(\boldsymbol{J}), ..., \alpha_n(\boldsymbol{J}), t) = \bar{W}(\boldsymbol{q}, \boldsymbol{J}) - \alpha_1(\boldsymbol{J}) t$$

This in turn implies that

$$p_\sigma = \frac{\partial \bar{S}}{\partial q_\sigma}$$
$$\bar{Q}_\sigma = \frac{\partial \bar{S}}{\partial J_\sigma}$$

[¶] the gradient w.r.t $\boldsymbol{q}$ is just $(\frac{\partial}{\partial q_1}, \frac{\partial}{\partial q_2}, ... \frac{\partial}{\partial q_n})$

[**] Remember that $\boldsymbol{P}$ is constant by assumption

Note that $\bar{P}_\sigma = J_\sigma$ and $\bar{Q}_\sigma = \bar{\beta}_\sigma$ are still constants. We next define the **angle variables**[††]:

$$w_\sigma \equiv \frac{\partial}{\partial J_\sigma} \bar{W}(\boldsymbol{q}, \boldsymbol{J}), \quad \sigma = 1, ..., n$$

We know that $\bar{\beta}_\sigma$ is a constant of motion, thus

$$\bar{\beta}_\sigma = w_\sigma - v_\sigma t, \quad v_\sigma \equiv \frac{\partial}{\partial J_\sigma} \alpha_1(\boldsymbol{J})$$

Thus

$$w_\sigma = v_\sigma t + \bar{\beta}_\sigma$$

v_σ is the *frequency* of the σ-th variable.

Poisson Brackets

I will not use the book's notation; instead I will use the standard notation, where the Poisson brackets are written as $\{\cdot, \cdot\}$. The Poisson brackets are defined as

$$\{f, g\} = \sum_\sigma \left(\frac{\partial f}{\partial q_\sigma} \frac{\partial g}{\partial p_\sigma} - \frac{\partial f}{\partial p_\sigma} \frac{\partial g}{\partial q_\sigma} \right)$$

The Poisson brackets are antisymmetric, thus

$$\{f, g\} = -\{g, f\}$$

The Poisson brackets of a function and the Hamiltonian describe the time-dependence of the function:

$$\{f, H\} = \sum_\sigma \left(\frac{\partial f}{\partial q_\sigma} \dot{q}_\sigma + \frac{\partial f}{\partial p_\sigma} \dot{p}_\sigma \right) = \frac{\mathrm{d} f}{\mathrm{d} t} - \frac{\partial f}{\partial t}$$

Thus a quantity, x, (without explicit time-dependence) is a constant of motion if

$$\{x, H\} = 0$$

Additionally Poisson brackets make for an easy test for whether variables are canonical. $\{\boldsymbol{p}, \boldsymbol{q}\}$ are canonical variables iff:

$$\{q_\alpha, p_\beta\} = \delta_{\alpha, \beta}$$

and

$$\{p_\alpha, p_\beta\} = 0 = \{q_\alpha, q_\beta\}$$

Some Poisson bracket rules:

$$\begin{aligned}
\{A + B, C\} &= \{A, C\} + \{B, C\} \\
\{AB, C\} &= \{A, C\} B + A \{B, C\} \\
0 &= \{A, \{B, C\}\} + \{B, \{C, A\}\} \\
&\quad + \{C, \{A, B\}\} \\
\frac{\mathrm{d}}{\mathrm{d} t} \{A, B\} &= \left\{ \frac{\mathrm{d}}{\mathrm{d} t} A, B \right\} \\
&\quad + \left\{ A, \frac{\mathrm{d}}{\mathrm{d} t} B \right\}
\end{aligned}$$

[††] which are canonically conjugate to the action variables