LIGHT-MATTER INTERACTIONS, STARS AND STELLAR SPECTRA

We measure the flux, F, of a star, which is related to its luminosity through the inverse square law:

$$F = \frac{L}{4\pi d^2}$$

Angles are usually measured using the small angle formula:

$$\alpha = \arctan\left(\frac{D}{d}\right) \approx \frac{D}{d}$$

where d is the distance to the object and D is the distance between the two points in the sky whose angle we are calculating. Astronomers usually use arcseconds as the unit for their angles, in which case you'd use

$$\alpha[\text{arcseconds}] = \frac{D[\text{AU}]}{d[\text{pc}]}$$

The spectrum of a star is approximately that of a black body, for which we have that

$$L = 4\pi R^2 \sigma_{\text{SB}} T^4$$

where the Stefan-Boltzmann constant is given by $\sigma_{\text{SB}} = 5.67 \cdot 10^{-8}\text{Wm}^{-2}\text{K}^{-4}$. Typical stellar temperatures are $3 - 100$kK: this is the temperature of the photosphere of a star, which is the "surface". The spectra of stars differ from perfect black-body spectra due to emission and absorption lines.

Generally we do not measure the total flux, but rather the flux at some wavelength, from where we define the flux per wavelength: F_λ, for which we have that $F_\lambda \Delta\lambda$ is equal to the total flux measured between λ and $\lambda + \Delta\lambda$. So if we apply some filter (band-pass), $\mathcal{T}_{\text{BP}}$, then this will weight the wavelength-dependent fluxes, as:

$$F_{\text{BP}} = \int \text{d}\lambda F_\lambda \mathcal{T}_{\text{BP}}(\lambda)$$

Stellar spectra

We primarily study stellar spectra in the interval $3500 - 7500$Å in the restframe of the star. We categorise stars depending on their temperature and hence their colour in the following order:

O B A F G K M

from hottest to coldest. The bluest stars have hydrogen and helium atomic and ionic absorption lines from their atmosphere, and the colder the star the bigger the molecules/atoms are that absorb light in the atmosphere. M-type stars have TiO and CaOH absorption lines for instance. We additionally categorise stars within each stellar type depending on their temperature adding 0 to the hottest and 9 to the coolest stars. A typical O-type star exceeds 30000K. There is a sharp decrease in flux around 3800Å, which is the Balmer jump, which corresponds to the transition $|2\rangle \rightsquigarrow |\infty\rangle$ of hydrogen (complete ionisation). Similar jumps occur at the Lyman and Paschen limits.

Another factor that affects the colour of stars is their metallicity: the metal-richer the redder. Therefore it can be beneficial to compare metallicities with the solar metallicity, this is done using [Fe/H] for instance, which is defined as

$$[\text{Fe/H}] = \log_{10}\left(\frac{n(\text{Fe})_\star n(\text{H})_\odot}{n(\text{H})_\star n(\text{Fe})_\odot}\right)$$

where $n(E)$ denotes the density of the element E.

The lives of stars

When stars are born they begin their lives on the main sequence: they begin by burning hydrogen, as this is the most abundant element in the universe and requires the least energy to burn. Naturally though the metallicity and hence the isochrone of recently born stars on a Hertzsprung-Russell diagram depends on the composition of the gas cloud in which the star is born.

The hydrogen is burnt primarily in the core, which produces helium. Once enough helium has formed in the core, the core contracts and hydrogen burning continues in a surrounding shell. This gives a "phase-transition"where the lifeline of a star on an HR-diagram experiences a discontinuity. Hydrogen burns in the shell until we reach another discontinuity: the core temperature and density increase, allowing the helium to undergo fusion too. After this stage it depends on the mass of the star:

1. $\mathcal{M} \leq 0.6\mathcal{M}_\odot$ have not left the main-sequence during the Hubble time
2. $0.6\mathcal{M}_\odot \leq \mathcal{M} \leq 2\mathcal{M}_\odot$ expand to hundreds of times their initial size. The expansion of the gas makes it cool: it becomes a red subgiant, and later a red giant.
3. $2\mathcal{M}_\odot \leq \mathcal{M} \leq 8\mathcal{M}_\odot$. Some become Cepheid variables, which pulsate so the luminosity varies as a function of time. Once the core has used up the helium the stars become red again: they become asymptotic giant branch stars. Rapid pulses

of expanding gas spew gases away from the star, and into the interstellar medium.

4. $\mathcal{M} \geq 8\mathcal{M}_\odot$ will also experience different fates. The lower mass part will explode as type-II supernovae: the core collapses which causes the outer shell to fall inward at $0.1c$, ending in a massive explosion. The remnant is either a neutron star or a black hole, depending on the mass of the star. Though specifically for $8\mathcal{M}_\odot \leq \mathcal{M} \leq 10\mathcal{M}_\odot$ the core may collapse before the core reaches iron, which can result in an electron-capture supernova or leave an oxygen-neon white dwarf.

For main sequence stars

$$L \sim \mathcal{M}^4$$

roughly.

Stellar photometry: the magnitude system

We define the apparent brightness as:

$$m = -2.5 \log_{10}(F)$$

which makes it convenient to compare brightnesses:

$$\Delta m = -2.5 \log_{10}\left(\frac{F_1}{F_2}\right)$$

for $\Delta m = 1$ we know that one star appears 2.5 times brighter than the other. Historically we used Vega as a reference: $m(\text{Vega}) = 0$.

Bandpasses

It is convenient to define a bandpass, BP, which is a filter through which we send the light before we calculate the flux. This helps us compare colours quantitatively. The flux in a bandpass is given by*

$$F_{\text{BP}} = \int \mathrm{d}\lambda F_\lambda \mathcal{T}_{\text{BP}}(\lambda)$$

and using this we can define the apparent magnitude in a bandpass. Comparing bandpasses tells you about colour: if B − V is positive it is redder than if it is negative.

Absolute magnitude

The apparent magnitude does not tell us how bright a star is, due to the inverse square law, therefore we can correct this effect by adding a term:

$$M = m - 5\log_{10}\left(\frac{d}{10\text{pc}}\right)$$

where M is the absolute magnitude.

Bolometric correction

Instead of measuring the total magnitude of a star you can measure it in a bandpass, and then add the bolometric correction to get the total magnitude. This bolometric correction is naturally a function of stellar type:

$$M_{\text{bol}} = M_{\text{V}} - \text{BC}$$

with this sign convention $\text{BC} = M_{\text{V}} - M_{\text{bol}}$.

Extinction

If there is no interstellar medium the flux and luminosity are related using the inverse square law:

$$F = \frac{L}{4\pi d^2}$$

however, if there is interstellar medium some of the light will be absorbed:

$$F_\lambda(x + \Delta x) = F_\lambda(x)\,(1 - \kappa_\lambda \Delta x)$$
$$\rightsquigarrow \quad F_\lambda(x) = F_\lambda(0)e^{-\kappa_\lambda(x-x_0)}$$

This exponential decay can be corrected for:

$$m_{\text{obs}} = m_{\text{int}} + A_{\text{mag}}, \quad A_{\text{mag}} > 0$$

the positive correction means that the object appears darker in the sky.

Dust reddens spectra (because blue light is scattered most), which can have two implications:

1. We infer a further distance due to the darker, redder spectrum
2. We infer a greater age, because metal-rich stars are redder

Distance Ladder

1. Radar: reflecting microwave off nearby objects to determine distances and geometry, as well as (radial) velocities.
2. Parallax: Measuring the position of an object with respect to the background at two opposite

* for a normalised bandpass filter

times in the year. This gives us a parallax angle, which tells us about the distance to the object:

$$d[\text{pc}] = \frac{1}{p[\text{arcsec}]}$$

Using Gaia this technique can be used up to kpc. Using non-Gaia data limits which stars in the galaxy have a parallax, which tells us that the Milky Way is greater than 500pc in diameter.

3. Photometric Parallax: By measuring the apparent and absolute magnitude of an object and subtracting them you get an expression for the distance. The absolute magnitude can be determined from its color ("all"main sequence M stars have the same luminosity for example)

4. Cepheid variables: Cepheid variables oscillatory frequency tells us about their luminosity (the brighter the longer the period), which makes it straight-forward to calculate the distance

5. Supernovae type-Ia: these explosions happen for $\mathcal{M} \approx 1.4\mathcal{M}_\odot$, which makes the explosions very similar. Therefore once we have calibrated one of these, using the lower rungs of the ladder we can use type-Ia supernovae as standard candles. We calculate L from τ.

6. Hubble-Lemaître Law: The HL law relates the velocity, and hence redshift to the distance to the object:

$$v = H_0 d = cz$$

where

$$z = \frac{\Delta\lambda}{\lambda_{\text{rest}}}$$

Proper motion

It is most natural to measure proper motion, μ, which is the angular motion in the sky, measured in arcseconds year^{-1} = as yr^{-1}. The relation between angular velocity and velocity gives us that

$$4.74\mu[\text{mas yr}^{-1}]d[\text{kpc}] = v_\text{t}[\text{km s}^{-1}]$$

where v_t denotes the tangential velocity.

Celestial coordinate system

The ecliptic plane is spanned by the vector that connects us and the sun, together with our orbital velocity. However the tilt of the earth changes as a function of time, therefore this coordinate system would need to be updated regularly.

Instead one can use the equatorial plane and define the azimuthal angle to be zero where the equatorial and ecliptic planes meet (vernal equinox). The zenith angle is referred to as declination, whereas the azimuthal angle is referred to as right ascension.

Components of the Milky Way

1. Thin Disk: $h_z = 300$pc, contains 95% of disk stars. The density of stars in the Milky Way as a function of height above the disk (towards the galactic north pole) can be fitted to a sum of exponentials with two different values of h_z : scale height. Thin disk stars have metal abundances between solar and about half-solar levels. Both disks orbit at about 200km s^{-1}

2. Thick Disk: $h_z = 1$kpc. The thick disk is metal-poorer with $0.1Z_\odot \leq Z \leq 0.5Z_\odot$. Generally the thick disk consists of stars that were in the thin disk before, but were kicked out, and hence the stars in the thick disk are older than those in the thin disk.

3. Bulge: The central bulge accounts for roughly 20% of the Milky Way's light. The Milky Way's bulge is barred, which is why it appears pear shaped in the sky. The density of stars rises towards the galactic centre, more so than predicted by the exponential law from above. The bulge extends further than the disks, but still rotates with the disks, approximately. The metal abundance is approximately that of the solar value, the average is slightly below, but there are stars with up to three times solar abundances. The rotation is slightly slower than the disk, at 100km s^{-1}.

4. Halo: The metal-poor halo consists of rogue stars and globular clusters. They do not have a preferred rotational direction: their dispersions are equal to their velocities and hence they often have eccentric orbits.

5. Globular Clusters: Globular clusters are metal-poor clusters of stars that all formed at about the same time: none of the clusters' stars formed less than a few Gyr ago, and therefore have very low metal-abundances. They live in the halo, far away from the disks. They have high densities of stars and are very bright, where the brightest, ω Centauri has $L \approx 10^6 L_\odot$ and contains about a million stars. The central region $r_c \approx 5$pc has an approximately constant density of stars. The exponential decay of density ceases at the truncation radius $r_t \approx 30$pc: stars beyond this are weakly bound and end up leaving the cluster.

6. Open Clusters: Contain up to several hundred stars and their luminosities are therefore $100 - 30000L_\odot$. Open clusters are often surrounded by gas and dust because they usually lie close to the disks: this makes them difficult to observe in the optical region. 95% of open clusters are younger than 1Gyr. Because most of open clusters' light comes from their brightest stars their colour is a good indicator for their age: just like stars: the older the redder.

7. Dark Matter: The measurement of the orbital velocity of stars and gas in the galaxy leads to a puzzling fact: the equation

$$\sqrt{\frac{G\mathcal{M}(< R)}{R}} = V(R)$$

underestimates the velocity at any given radius: there is more mass in galaxies than what is visible: this is dark matter.

At the centre of the Milky Way we find *Sagittarius A**, which is a compact radio source (black hole), with $\mathcal{M} \sim 4 \cdot 10^6 \mathcal{M}_\odot$. It is $\sim$ 8kpc away. Due to the dust between us and the Sgr A* we must observe it with radiofrequency and infrared.

Velocity Dispersion

The velocity dispersion, σ, is defined as

$$\sigma = \sqrt{\langle v_z^2 \rangle - \langle v_z \rangle^2}$$

which is equivalent to the standard deviation of the velocity. Generally σ increases as a function of age.

Galactic rotation

The radial and tangential velocities (with respect to earth) of stars nearby can be calculated using Oort's formulae:

$$V_r = dA\sin(2\ell)$$
$$V_t = d(A\cos(2\ell) + B)$$

where A and B are Oort's constants. They can be derived from the tangent point method.

SPIRAL GALAXIES

Components:

1. Spiral Arms: young stars– stars form in the spiral arms
2. Central Bulge: gas poor with high density of stars
3. Halo
4. Dark Matter

Spiral galaxies are classified using the Hubble sequence. `S0` have the biggest bulge and barely any spirals, and then `Sa`, `Sb`,... have increasing spirals and decreasing bulges. If the bulge is barred we add `B` after the `S`, so `SB0` is a `S0` galaxy with a barred bulge. `SB0` and `S0` galaxies are lenticular galaxies.

Dim galaxies have not used their gas yet, whereas bright galaxies have formed many stars that radiate.

Star formation requires cold gas, so that the contraction can be sufficient. The spiral arms are where stars are born primarily, for spiral galaxies. The central bulge has used up all of its gas, and hence there is barely any stellar formation in that region.

Spiral Arms

The arms are not rigid, otherwise they would fold on themselves as time goes. It is not known what mechanism creates the spiral arms, but the leading theories are density waves and kinematic spirals that form due to elliptic orbits of stars about the galactic centre.

The arms almost always *trail*, which means that the tip points backwards with respect to the galactic rotation.

Surface brightness profile

Spiral galaxies' surface brightnesses decay exponentially as a function of radius, just like the number density does:

$$I(r) \sim e^{-R/h_R}, \quad h_R : \text{ scale length}$$

the scale length is normally on the order of 10kpc.

Cool gas in the disk

Cold neutral hydrogen, `HI`, emits 21cm radiation due to the hyperfine-splitting of the ground state. This radiation can be measured everywhere in space, and because of its long wavelength is very useful: long wavelengths are scattered less by gas and dust. This radiation is a very good candidate to study disk velocities with.

Rotation curves

Spider diagrams show a galaxy as an ellipse (2d projection of a round galaxy seen from some angle). Using the 21cm radiation and its Doppler shift we can calculate the velocity of the disk: one side of the disk moves towards us, the other away. In the middle there is a region that only travels tangentially to us – we cannot use Doppler to measure this velocity. Thus the isotachs will look like a spider: this is a spider diagram.

A rotation curve shows the tangential velocity along the major axis of the spider diagram. Therefore it will look like the derivative of a gaussian (approximately).

The tangential velocity grows initially, at small radii, but *flattens* out at some characteristic radius: this is not what we expect. If the galaxy rotated like a solid object, then the rotation curve would be linear $v = \omega r$. If it were like the solar system, then it would go like $r^{-1/2}$. This leads to the hypothesis that there is invisible mass in the galaxy: dark matter. Most of the gravitating mass of galaxies is dark matter.

Surface Brightnesses

While the flux decreases with radius the flux density (surface brightness) will be constant, for non-point sources. This doesn't hold for intergalactic distances, because then the expansion of the universe plays a role, multiplying by $(1+z)^{-4}$. The light distribution of disks is approximately exponential:

$$I(R,z) = I(0)e^{-|z|/h_z}e^{-R/h_R}, \quad h_z \approx 0.1h_R$$

However, this does not hold for the bulge: here the surface brightness deviates and peaks at zero distance from the centre. The surface brightness depends on the inclination.

Sersic Profile

The Sersic profile attempts to model this deviation from exponential:

$$I(r) = I_e \exp\left(-b\left(\left(\frac{r}{r_e}\right)^{1/n} - 1\right)\right)$$

where n is a parameter and r_e is some effective radius. For $n = 1$ this reduces to the exponential profile. For $n = 4$ we recover De Vaucouleurs law, which applies to elliptical galaxies. Early type† galaxies have greater vertical height scales.

Tully-Fischer Relation

The Tully-Fischer relation gives a relationship between the maximal tangential velocity of a galaxy and its luminosity:

$$L \sim V_{max}^{\alpha}$$

† Elliptical and `S0` galaxies.

$\alpha = 4$ normally, but $\alpha_B \approx 3$ (B-band). This‡ makes intuitive sense, because a more massive galaxy will tend to have more stars, and hence will be brighter. The higher mass will in turn also force the orbital speeds to be higher, because of the virial theorem. We can use this to measure distances to galaxies, once we have calibrated.

The Tully-Fischer relation is often quoted as

$$\frac{L_I}{4 \cdot 10^{10} L_{I,\odot}} = \left(\frac{v}{200\text{km s}^{-1}}\right)^4$$

It can be used for distance measurements.

ELLIPTICAL GALAXIES

Normal elliptical galaxies fill in the classes `E`, `gE` and `cE`, where the last two are giant and compact elliptical galaxies respectively. The absolute magnitude of `gE` is $M < -20$ and generally elliptical galaxies fall within $-23 < M < -15$. Additionally there are dwarf ellipticals, `dE`, with even lower surface brightness and dwarf spheroidals, `dSph`, that are even darker. Finally `cD` galaxies are extremely bright. `dE` have more compact cores than `dSph`.

Elliptical galaxies are classified with `EN` where

$$N = 10\left(1 - \frac{b}{a}\right)$$

with a and b the major and minor axis respectively. We have `E0,E1,...,E7`, galaxies with greater ellipticity than $\varepsilon = 0.7$ are dynamically unstable. We further put them in three groups:

$$\begin{aligned} a = b < c &\quad \rightarrow \quad \text{oblate spheroid} \\ a < b = c &\quad \rightarrow \quad \text{prolate spheroid} \\ a \neq b \neq c &\quad \rightarrow \quad \text{triaxial} \end{aligned}$$

We see the difference between triaxials and other types using isophote twist, where the major axes of the isophotes rotate with respect to each other. If an elliptical has more light along the major axis than the other directions it is referred to as "disky", if it has more light along the four direction off of the major axes it is "boxy".

Key properties of elliptical galaxies:

‡ the fact that these quantities are related

1. Smooth L distribution
2. No stellar formation $\rightsquigarrow$ older populations
3. Major and minor axes: We define the ellipticity, ε, and class `EN`. This may just be an artefact of projection!
4. Random motions: slow/no rotation. Dispersion is comparable to orbital velocity
5. Not much dust: redness is due to age of stars!

We classify them further using the Sersic index, n, as well as R_{25} which is the 25-th isophote. Additionally the ellipticity and velocity dispersion in all three dimensions. `dE`s tend to have a Sersic index near 1, whereas normal ellipticals often have $n \approx 4$, which recovers De Vaucouleurs.

Largest and brightest ellipticals have the largest core radii, but a relatively low central surface brightness: distributed stars in centre. Though many ellipticals have a central cusp where the brightness grows rapidly, as predicted with De Vaucouleurs, this cannot always be seen, because of seeing.

Ultra-compact ellipticals may be the bulges from spiral galaxies, where the disk was ripped off.

Stellar interactions will cause elliptical galaxies to become spherical, however this will happen on a time scale greater than the Hubble time, due to the low density of stars.

The lack of star formation implies there isn't very much cold gas, however, there is hot gas, which can be seen in the x-ray region, for example.

Fundamental plane

Using the effective radius from the Sersic Profile, we get that

$$R_e \propto \sigma^{1.2} I_e^{-0.8}$$

taking the logarithm of this you get the equation for a plane, which is known as the fundamental plane.

GALACTIC DYNAMICS

The interactions between stars can be put into two groups:

1. Strong encounters: changes the trajectory abruptly (and significantly)
2. Weak encounters: slow, gradual changes.

The potential that a star experiences can thus be thought of as a smooth macroscopic well, the weak encounters, together with delta functions, at the position of stars, which corresponds to the strong encounters.

Let us define a characteristic energy scale for encounters:

$$U = \frac{GmM}{b}$$

That is (minus) the gravitational energy at closest encounter. From this quantity we can define weak and strong encounters: for strong encounters this energy is comparable to or greater than the kinetic energy, whereas it is smaller for weak encounters. Thus, we can define a strong encounter distance:

$$r_s = \frac{2Gm}{v^2}$$

which implies we can change the condition to $r < r_s$ for strong encounters. For the sun $r_{s,\odot} \approx 1\text{AU}$.

For weak encounters $r > r_s$, but ideally $r \gg r_s$. The angle between initial and final trajectories, α, must be sufficiently small: the trajectory is barely changed during weak encounters. This approximation is equivalent to the impulse approximation, which is that the interaction is rapid compared with the orbital time.

The component of the force that changes the *direction* of the velocity can be written as

$$F_\perp = \frac{GmMb}{\left(b^2 + v^2t^2\right)^{3/2}}$$

which gives the change in perpendicular velocity as:

$$\Delta v_\perp = \frac{1}{M} \int_{-\infty}^{\infty} \mathrm{d}t \frac{GmMb}{(b^2 + v^2t^2)^{3/2}} = \frac{2Gm}{bv}$$

This gives us an expression for the angle, α:

$$\alpha \approx \frac{2Gm}{bv^2} = \frac{\Delta v}{v}$$

where the small angle approximation holds by assumption. Note that the integral above will barely change if we change the limits to $\pm\frac{b}{v}$ (70%) and $\pm\frac{b}{2v}$ (45%).

The change in kinetic energy of the star is:

$$\Delta T = \frac{1}{2} M \left(\Delta v_\perp^2 + 2\boldsymbol{v} \cdot \Delta \boldsymbol{v}_\perp\right)$$

this quantity can be positive or negative. This shows that (weak) encounters dissipate energy through the galaxy, which gives relaxation.

By considering a cylindrical cutout of the galaxy, following the trajectory of a star, we can estimate the expected value of the change in kinetic energy:

$$\begin{aligned}\langle \Delta v_\perp^2 \rangle &= \int_{b_{min}}^{b_{max}} n 2\pi b(vt) \left(\frac{2Gm}{bv}\right)^2 \mathrm{d}b \\ &= \frac{8\pi G^2 m^2 nt}{V} \ln\Lambda, \quad \Lambda \equiv \frac{b_{\max}}{b_{\min}}\end{aligned}$$

Generally

$$\begin{aligned} b_{\max} &\sim \text{size of system} \\ b_{\min} &\sim \text{smallest distance} \sim r_s \end{aligned}$$

To estimate relaxation times, let us calculate how long it takes for $\langle \delta v_\perp^2 \rangle$ to equal v^2:

$$t_{\text{relax}} = \frac{v^3}{8\pi G^2 m^2 n \ln\Lambda}$$

This time scale is different, depending on the size of the system:

1. Solar neighbourhood: $t_{\text{relax}} \sim 10^{13}\text{yr}$
2. Globular clusters: $t_{\text{relax}} \sim 5\text{Gyr}$
3. Open clusters: $t_{\text{relax}} \sim 50\text{Myr}$

In contrast, taking the crossing time to be $t_{\text{cross}} \approx \frac{R}{v}$:

1. Solar neighbourhood: $t_{\text{cross}} \sim 30\text{Myr}$
2. Globular clusters: $t_{\text{cross}} \sim 0.5\text{Myr}$
3. Open clusters: $t_{\text{cross}} \sim 5\text{Myr}$

We can estimate the ratios:

1. Solar neighbourhood: $t_{\text{relax}} \sim 10^9 t_{\text{cross}}$
2. Globular clusters: $t_{\text{relax}} \sim 10^4 t_{\text{cross}}$
3. Open clusters: $t_{\text{relax}} \sim 10 t_{\text{cross}}$

The relaxation is only relevant for open clusters! This implies that there is a minimal radius for which we can trust numerical and theoretical analyses.

In the opposite limit we see very few effects: strong encounters are rare. $t_s \sim 40 t_{\text{relax}}$, always.

Mass segregation

The mean kinetic energy gives us, very roughly:

$$\langle v^2 \rangle = \frac{3k_B T}{m}$$

therefore we expect that large masses will have an average speed that is smaller than light masses. This implies that heavy bodies will tend to fall inwards. This is the case in relaxed clusters: the centre has heavier stars and the outer parts have more lighter stars.

Evaporation

The Maxwell distribution of velocities implies that there always will be a tail ($\sim 1\%$) of stars that have a velocity that is greater than the escape velocity. These stars will escape after about one relaxation time, therefore after $\sim 100 t_{\text{relax}}$ a body will have evaporated.

$$t_{\text{evaporation}} \sim 100 t_{\text{relax}}$$

This implies that it is only open clusters in which evaporation occurs on a time scale smaller than the Hubble time.

Tensor Virial Theorem

Consider the quantity

$$I_{ij} = \sum_\alpha m_\alpha r_\alpha^i \cdot r_\alpha^j$$

The second derivative hereof:

$$\frac{\mathrm{d}^2 I_{ij}}{\mathrm{d}t^2} = \sum_\alpha m_\alpha \left(2\dot{x}_\alpha^i \dot{x}_\alpha^j + \ddot{x}_\alpha^i x_\alpha^j + x_\alpha^i \ddot{x}_\alpha^j\right)$$

using the expression for Newtonian gravity we can find that

$$\frac{1}{2}\frac{\mathrm{d}^2 I_{jk}}{\mathrm{d}t^2} = 2T_{jk} + V_{jk}$$

In equilibrium the moment of inertia will be constant[§]:

$$2T_{jk} = -V_{jk}$$

which is the virial theorem in tensor form. This leads naturally to the distinction between rotation supported and dispersion supported galaxies, which depends on which of the velocity terms is the main contributor to the kinetic term in the virial theorem. Generally dispersion supported galaxies have no reason to be flat.

[§] This is a sufficient but not necessary condition – we can require I only changes linearly in time!

Thus elliptical galaxies are dispersion supported, whereas spiral galaxies are rotation supported. For *dispersion* supported galaxies we have:

$$\frac{L_V}{2\cdot 10^{10}L_{V,\odot}} = \left(\frac{\sigma}{200\text{km s}^{-1}}\right)^4$$

which is the **Faber-Jackson** relation. Note the similarity to the Tully-Fischer relation. This can be used for distance measurements.

For fast rotating galaxies we have that

$$\frac{V_{\max}}{\sigma} = \sqrt{\frac{\varepsilon}{1-\varepsilon}}$$

GALAXY GROUPS & CLUSTERS

Here are some fundamental differences between galaxy clusters and groups:

1. Groups: loose connection of up to a few dozen galaxies. The weak binding energy allows for spiral galaxies to exist
2. Clusters: 100s to 1000s of galaxies, extending up to 10Mly

An interesting fact is that galactic clusters are the largest structures that are gravitationally bound – anything larger will be affected by the expansion of the universe.

Statistics

The average density of galaxies in the universe is

$$\langle n\rangle = 0.1 \text{ galaxy Mpc}^{-3}$$

This means that the probability of finding 10 galaxies within a cubic megaparsec is $\sim 10^{-6}$ and of finding 100 is $\sim 10^{-160}$. This implies that it is extremely unlikely that groups and clusters exist due to statistical variations. Therefore they must be the result of gravitational attraction.

Galactic Groups

The Milky way, Andromeda and a few satellites form the *local group*. We classify groups as sparse, compact and fossil, depending on the size of stellar populations of the galaxies:

1. Sparse: Only a few galaxies
2. Compact: high density
3. Very old (late stage), massive galaxies, probably merged many a time.

Galactic groups primarily consist of spiral galaxies with few ellipticals. The galactic dispersions are $\sigma < 500\text{km s}^{-1}$, which is comparable to stellar dispersions!

There are frequent and violent galactic interactions: mergers typically happen in groups. When two galaxies merge the gas is affected much more than anything else: gas is spewed in tails and through the centre. This process can strip galaxies of their gas abruptly stopping their stellar production.

Galactic Clusters

For example Virgo and Coma. These primarily consist of elliptical galaxies and the galaxies have greater dispersions than in groups: $\sigma \sim 1000\text{km s}^{-1}$ ($> 500\text{km s}^{-1}$). Typical masses are $10^{14}\mathcal{M}_\odot \le \mathcal{M} \le 10^{16}\mathcal{M}_\odot$.

All galactic clusters have x-ray gas ($3\text{MK} < T < 200\text{MK}$), which is their main baryonic constituent (up to ten times more gas than stars). They also have a large amount of dark matter: $\mathcal{M}/L \sim 180 - 300\mathcal{M}_\odot/L_\odot$ (up to 85% of mass is dark matter).

The x-rays are emitted during Bremsstrahlung processes of electrons in the centre.

Finding groups and clusters

1. High angular density (all at about the same radial distance)
2. x-ray telescopes
3. Friends of friends algorithm

The electrons in the centres of clusters inverse-Compton scatter the cosmic background radiation. This means that dark spots in the CMB can be due to clusters.

Intra cluster medium

The gaseous medium within (the centres of) clusters radiates x-ray radiation, the luminosity of which is temperature dependent:

$$L_{\text{x}} \sim n^2 T^{1/2}$$

This radiation cools the gas. We can estimate the cooling rate

$$t_{\rm cool} \sim \frac{3nk_BT}{3\cdot 10^{-27}n^2T^{1/2}} \approx 14\left(\frac{10^{-3}{\rm cm}^{-3}}{n}\right)\left(\frac{10^7{\rm K}}{T}\right)^{1/2}{\rm Gyr}$$

typically $t_{\rm cool} \sim 14$Gyr. This makes sense, because we can still see the x-rays from the central gas. However, n grows in the centre, which can shorten the cooling time and produce cool-core clusters.

Typical temperatures of the central gas are $10-100$MK.

How do we measure cluster masses?

1. Velocity dispersion:

$$\mathcal{M} = 7\cdot 10^{14}\mathcal{M}_\odot \frac{1}{\alpha}\left(\frac{\sigma}{1000{\rm km\ s}^{-1}}\right)^2\left(\frac{r_c}{3{\rm Mpc}}\right)$$

2. x-ray measurement: The thermal pressure is what keeps the gas in hydrostatic equilibrium therefore we can set the pressure gradient equal to the gravitational force, which gives us

$$\mathcal{M}(<r) = \frac{k_B}{m_{\rm particle}}\frac{r^2}{G\rho(r)}\frac{{\rm d}(-\rho T_x)}{{\rm d}r}$$

   This includes all massive objects in the cluster. This gives us that there is $8\cdot 10^{13}\mathcal{M}_\odot$ within 200kpc of the centre of the Coma cluster.

3. Gravitational lensing:

$$\mathcal{M}(<\theta_{\rm E}) = \left(\frac{d_{\rm lens}}{100{\rm Mpc}}\right)\left(\frac{\theta_{\rm E}}{1''}\right)^2 10^{10}\mathcal{M}_\odot$$

   We further distinguish between

   a. Strong lensing: background image is seen numerous times and image is distorted
   b. Weak lensing: only see the background image once, but there is elliptical distortion

Structure of clusters

We define two radii: the virial radius, R_v, to which the virial theorem holds. Most of the mass is within R_v. The second radius, R_I, is the radius where the expansion of the universe begins to dominate: there begin to be a force that pushes outwards. Generally $R_I = 4R_v$. The different mass measurement techniques have different limitations:

1. Strong lensing & x-ray emission: $\sim R_v$
2. Weak lensing & velocity dispersion: $\sim R_I$

We consistently measure that there is more gravitational mass than there is luminous mass.

Bullet cluster

Two clusters collided and the gas interacts. Due to ram pressure the gas lags behind the galaxies. However the collisionless dark matter continues unhindered, which can be seen through gravitational lensing. This phenomenon forced us to disregard many opposition theories to dark matter: now dark matter is the most common theory.

Dynamic friction

When a heavy object passes through a medium of light objects, it will accelerate the lighter objects and in doing so lose energy and momentum. It can be shown that this effect is greater for large masses than for small masses, as we saw for weak encounters. This is the basic principle behind dynamic friction: heavy objects decelerate and sink inwards.

$$\frac{{\rm d}v}{{\rm d}t} = -\frac{4\pi G^2(M+m)nm\ln\lambda}{v^2}, \quad \lambda \equiv \frac{b_{\rm max}}{b_{\rm min}}$$

For media with random motion $\dot{v} \propto -v$. The velocity oscillates and decays, falling inwards.

When galaxies fly by each other their dispersions increase, which causes them to puff up. If their relative velocities are low enough they can in fact collide and merge, therefore galaxy groups have the perfect conditions for mergers (this is also where they are often seen observationally). During the merger process gas will also be compressed due to the collision. This can cause an abrupt stellar formation period: star burst galaxy. This strong stellar formation changes the hydrostatic conditions, and can create a starburst galaxy.

Mergers

Galactic collisions usually end as elliptical galaxies. Fossil groups is the final evolutionary step of galactic groups. These are big ellipticals with possible satellites – they are called fossils because of the low stellar formation rate.

Harassment is an effect of strong encounters between galaxies. The strong encounter removes the disk (or outer part) of the galaxy, because of the close encounter. If we begin with a spiral galaxy, and harass it, we will end with a (compact?) elliptical galaxy, which corresponds to what used to be the bulge.

Ram-pressure stripping refers to the process of removing gas from galaxies as they move through the intergalactic medium due to friction. Ram-stripped galaxies will leave tails, just like comets do.

Mergers, ram-pressure and harassment transform disky galaxies to gas-poor elliptical galaxies. This process of dynamic evolution from spiral to ellipticals is described by the **Butcher-Oemler effect**. It is in fact observed that higher redshift clusters and groups have greater spiral populations

Morphology density relation: the higher density the more ellipticals

Roughly $1/2$ of all galaxies are in groups and clusters

ACTIVE GALACTIC NUCLEI

Black holes

Black holes are created when the density of an object is sufficient to create a singularity in spacetime. This strong gravitational attraction prevents everything, even light, from escaping within the Schwarzschild radius:

$$R_s = \frac{2G\mathcal{M}}{c^2} = 3\text{km}\frac{\mathcal{M}}{\mathcal{M}_\odot}$$

at this radius c is the escape velocity. The masses of black holes vary over numerous orders of magnitude, therefore we categorise them as following:

1. Stellar black holes: $\sim (3-100)\mathcal{M}_\odot$
2. Intermediate black holes: $\sim 10^4\mathcal{M}_\odot$
3. Supermassive black holes: $\sim (10^6-10^{10})\mathcal{M}_\odot$

How are they formed?

blue supergiant $\rightsquigarrow$ supernova $\rightsquigarrow$ black hole

primarily.

We recently measured a black hole merger, using the radiated gravitational waves. This was measured using LIGO.

Supermassive black holes

We study the black holes at the centre of galaxies, as these tend to be supermassive. Specifically we have studied Sagittarius A*, which lies in the centre of the Milky Way. We estimate the mass of Sgr A* using the virial theorem:

$$M_{\text{BH}} = \frac{Rv^2}{G}$$

Active Galactic Nuclei

We define active galactic nuclei as compact galactic centres that radiate at higher than average rates. Active galactic nuclei emit a non-stellar spectrum, which is too intense to be due to fusion. The short time-variability shows that the emitting region is compact, and therefore the source of radiation must be more efficient than fusion.

Estimating the efficiency of the process to be 10% (that is 10% of the accreted mass is converted to radiation) we get a mass increase of the black hole of

$$\dot{\mathcal{M}} = 0.2\mathcal{M}_\odot \text{ yr}^{-1}$$

Eddington Luminosity

An upper limit for the luminosity of an active galactic nuclei given its mass is the **Eddington Luminosity**, which assumes the hydrostatic equilibrium is achieved through this intense radiation:

$$\frac{\sigma L_{\text{Edd}}}{4\pi c} = G\mathcal{M}(m_\text{e} + m_\text{p})$$

This gives us

$$L_{\text{edd}} \approx 3 \cdot 10^4 \left(\frac{\mathcal{M}}{\mathcal{M}_\odot}\right) L_\odot$$

This gives us the following prediction:

1. Seyfert Galaxy:

$$\mathcal{M}_{\text{BH}} \sim 10^7\mathcal{M}_\odot \quad \rightsquigarrow \quad L_{\text{BH}} = 10^{11}L_\odot$$

2. Quasar

$$\mathcal{M}_{\text{BH}} \sim 10^9\mathcal{M}_\odot \quad \rightsquigarrow \quad L_{\text{BH}} = 10^{13}L_\odot$$

However, the Eddington luminosity overestimates the luminosity, in reality:

$$\frac{L_{\text{Seyfert}}}{L_{\text{Edd}}} \sim 0.3\% - 3\%$$
$$\frac{L_{\text{Quasar}}}{L_{\text{Edd}}} \sim 0.3\% - 33\%$$

Structure of Active galactic nuclei

The spectrum from a active galactic nuclei contains emission at all wavelengths, with a peak in the optical/UV frequencies. Some have a radio-frequency signal (radio-loud) and some barely show any radio-frequency signals (radio-quiet). There are two bumps: one in the infrared region and one in the optical/UV. Additionally we see Doppler line-broadening.

There are also forbidden transitions, which suggests that the density of gas is low. The spectrum is time-dependent.

The accretion disk is a pseudo-2d surface which orbits the central black hole. The friction between particles heats the disk, which first of all causes continuum emissions, and second of all causes the accretion disk to go inwards. The potential energy decreases. At the centre we see short wavelength emissions.

Beyond the disk is an accretion torus (doughnut) which can block the broad line region's emission, which means we only see the narrow lines. The torus emits in the infrared region.

Active galactic nuclei components:

1. Supermassive black hole: $R_s \sim (0.02 - 20)\text{AU}$. Radio waves come from jet emission, and x-ray photons come from scattering at the corona.
2. Accretion disc: $1 - 2\text{light days}(200 - 300\text{AU})$. UV/optical emission, due to the thermal energy. The temperature of the disc depends on the distance from the centre (decreasing function of r)
3. Broad line region: $1000 - 3000\text{AU}$. Broadened UV emission lines.
4. Narrow line region: $100\text{pc} - \text{kpc}$. Narrow optical emission lines.
5. Jets: $\text{pc} - \text{kpc}$

The broad line region's radius can be estimated using reverberation mapping.

Unification theory

The unification theory proposes that the different types: Seyfert galaxies, radio galaxies and quasars are all the same thing, just viewed from different angles.

The greater the central black hole, the greater the bulge of galaxies.

LARGE SCALE STRUCTURES

We can create a 3D map of the universe, using right ascension, declination and redshift. Unfortunately the redshift is not a perfect tool to calculate distances with, because the observed redshift is due to the cosmic velocity, as well as a peculiar velocity.

We see that the universe is not homogeneous, there are regions with high galactic densities, and regions that essentially are void of galaxies. This is naturally due to the gravitational attraction of galaxies, and the effective "repulsion"of voids.

Naturally the Milky Way disk makes it difficult to see past it: zone of avoidance. We can see through it using 21cm radiation, then using the redshift to determine whether the observed source is in the disk or behind it.

We talk about three main constituents:

1. Voids: large regions that essentially are void of luminous mass
2. Walls: regions that have many luminous objects
3. Filaments: elongated regions that combine walls

As mentioned previously redshift is not a perfect tool for distance measurement. We use the Hubble-Lemaître law:

$$d = \frac{cz}{H_0}, \quad z \equiv \frac{\lambda_{\text{observed}}}{\lambda_{\text{emit}}} - 1$$

However cz must be greater than the dispersion. The Hubble-Lemaître Law is definitely not valid nearby, when

$$v_r = cz = H_0 d + v_p$$

we refer to v_p as the peculiar velocity. But luckily we can determine the distance to close objects using other rungs on the distance ladder, such as the Tully-Fisher relation.

Surface- Brightness fluctuations

The fluctuation of light at each pixel goes like d^{-1}, which gives us distances, once we have calibrated.

Pie-charts/ wedge diagrams

Plotting observed sources in polar coordinates, with z and an angle (right ascension or declination) will give areas of high and low density. Consider a cluster of galaxies and in front of and behind this there are groups/ galaxies moving in towards it. If the inward velocity is comparable to cz then the objects behind seem to have a smaller redshift, and the object in front of seem to have a higher redshift, which may put them on top of each other:

> Walls may appear thicker due to peculiar velocities.

Red galaxies form less homogeneous shapes on the wedge-diagrams than blue galaxies. For example in the Coma cluster the spiral galaxies are uniformly distributed, but ellipticals are mostly in the middle¶

Galaxy surveys are biased

We naturally have to choose some lower limit for the luminosity, to avoid noise. Therefore we see fewer galaxies at high redshift. Thus this is not because there are fewer galaxies at high redshift, but because we cannot include them in our measurements, until we increase our accuracy.

Additionally if an object is too compact we assume it is a star: it is not included in galaxy surveys.

Clumping & two-point correlation

Let us describe the distribution of luminous objects in our measurements as

$$\Delta P = n^2(1+\xi)\Delta V_1 \Delta V_2, \quad \xi = \left(\frac{r}{r_0}\right)^{-\gamma}$$

where r_0 is the correlation length. This tells us about the probability of finding a galaxy a distance r from another galaxy. This is a measure of the deviation of the universe from perfect homogeneity, therefore we'd expect this to be a function of redshift (the more redshift the smaller $|\xi|$)

This naturally assumes (cylindrical) symmetry, therefore it is good for describing clusters, but bad for describing walls and filaments.

¶ It appears this is not mass segregation, but rather related to the fact that high densities increases the probability of encounters, which through the Butcher-Oemler effect converts spirals into ellipticals.

CLOSED BOX MODEL

Begin with an isolated box with gas, which has mass M_G. The gas forms stars, which through supernova explosions enriches the gas with heavy elements.

We assume:

1. No initial metallicity
2. Homogeneity (always, also right after a supernova explosion)
3. Instantaneous return of metals from high mass stars (very short lifetimes)

We need to include $M_\star$ and M_h; the mass of gas locked up in stars and the mass of heavy elements respectively. This defines the metallicity function:

$$Z(t) = \frac{M_h(t)}{M_g(t)}$$

The birth of stars implies:

$$\mathrm{d}M_g = -\mathrm{d}M_\star \quad \rightsquigarrow \quad \mathrm{d}M_h = M_g \mathrm{d}Z + Z\mathrm{d}M_g$$

The term $M_g \mathrm{d}Z$ is proportional to the change (increase) in the metallicity, therefore this term can be represented by $p\mathrm{d}M_\star$, where p tells us about the yield from supernova explosions. Hence

$$(p - Z)\mathrm{d}M_\star = \mathrm{d}M_h$$

Thus

$$\mathrm{d}Z = \frac{p\mathrm{d}M_\star - Z(\mathrm{d}M_\star + \mathrm{d}M_g)}{M_g}$$

The assumption of the closed box model is that nothing enters or leaves the system, therefore $\mathrm{d}M_\star + \mathrm{d}M_g = 0$.

$$\mathrm{d}Z = \frac{p\mathrm{d}M_\star}{M_g} = -p\,\mathrm{d}\ln M_g$$

Thus

$$Z(t) = Z_0 + p\ln\left(\frac{M_g(t=0)}{M_g(t)}\right)$$

Inverting

$$M_g(<Z) = M_g(0)\exp\left(-\left(\frac{Z - Z_0}{p}\right)\right)$$

But $M_g(<Z) + M_\star(<Z) = M_g(0)$, hence

$$M_\star(<Z) = M_g(0)\left(1 - \exp\left(-\left(\frac{Z - Z_0}{p}\right)\right)\right)$$

In the solar neighbourhood: $p \approx 0.74 Z_\odot$, which implies that

$$\frac{M_\star(< 0.25 Z_\odot)}{M_\star(< Z_\odot)} \approx 0.4$$

40% of the stars near the sun should have less than $0.25 Z_\odot$. However our observations show that $1/4$ have less than a quarter of the solar iron abundance, and only one has less than a quarter of the abundance of oxygen. This issue is known as the **G-dwarf problem**, and it implies that we cannot treat the disk as a closed box.

The closed box model works well for the Milky Way bulge. Halo stars have a lower metallicity than the closed box model predicts: this is possibly because these are not heavy enough to hold on to metals (they escape into the intergalactic medium)

LYMAN-α ABSORPTION: INTERGALACTIC MEDIUM

Intergalactic gas can only be observed indirectly: through Lyman-α absorption. Lyman-α absorption corresponds to the doublet

$$\begin{aligned} 1s_{1/2} &\rightsquigarrow 2p_{3/2} \\ 1s_{1/2} &\rightsquigarrow 2p_{1/2} \end{aligned}$$

transitions. The splitting between the energies is small, so it's normally just quoted as $\lambda_{\text{rest}} = 1216\text{Å}$. We see that the density of Lyman-α absorbers increases as $(1+z)^{5/2}$, which implies that the density of neutral hydrogen was higher in the past.

At high z you find Lyman-α forests. These are absorption lines blueward of the rest-frame Lyman-α transition. These are due to neutral hydrogen clouds between the target and us, that absorb through Lyman-α excitation (amongst other transitions) at different redshifts, resulting in a forest of absorption lines.

The Lyman-α forest implies that we expect the average flux bluewards of $\lambda_{\text{rest}} = 1216\text{Å}$ to be lower than the average flux redwards of the rest Lyman-α wavelength. This is the **Gunn-Peterson Effect.** There will be some parts of the spectrum that are completely absorbed, these are referred to as Gunn-Peterson troughs.

There is hydrogen everywhere, and yet we see parts of the continuum spectrum of quasar spectra, which implies that hydrogen is highly ionised.

Close to the quasar there is a region where the hydrogen is ionised, `HII`. The radius of this region can be up to a few Mpc, therefore the relative redshift is measurable. In this region there are fewer Lyman-α transitions, which implies that this region has a decreased Lyman-α forest. Thus there will be a narrow region bluewards of the quasars Lyman-α rest-wavelength, where there are measurable fluxes, just before the Gunn-Peterson effect begins to take effect.

The absorption of the intergalactic medium depends on its (neutral) column density:

1. $n(\texttt{HI}) > 2 \cdot 10^{20}\text{cm}^{-2}$: Damped Lyman-$\alpha$ cloud

   In this limit the neutral hydrogen is optically thick, with prominent damping wings. This thick gas is referred to as a damped Lyman-α cloud.

2. $n(\texttt{HI}) \sim 2 \cdot 10^{17}\text{cm}^{-2}$: Lyman-limit cloud

   A Lyman-limit cloud is dense enough to absorb almost all photons that have enough energy to ionise hydrogen. This means the flux drop rapidly at $\lambda_{\text{rest}} = 912\text{Å}$. Thus a large amount of the Lyman-limit clouds is ionised hydrogen.

3. $n(\texttt{HI}) < 2 \cdot 10^{17}\text{cm}^{-2}$: Lyman-$\alpha$ forest.

   The low density implies that there will be a very narrow line width. Thus the Lyman-α forest consists of very many narrow absorption lines from different redshifts.

We can estimate the density of damped Lyman-α clouds in the universe from measurements, this is an increasing function of z, which once again points towards the fact that there was more neutral hydrogen in the past. This phenomenon cannot purely be explained by stellar formation, and therefore we hypothesise that there is more ionised hydrogen now than there was at higher z.

METALLICITY

Regions with high metallicity include:

1. Bulge
2. (Thin) disk ($(0.2-2)Z_\odot$ in thin disk and $(0.1-0.2)Z_\odot$ in thick disk)
3. Spiral arms

regions with low metallicity include:

1. Halo ($(0.01-0.3)Z_\odot$ with few stars with metallicities as low as $10^{-4} Z_\odot$)

2. Globular clusters

3. large redshift galaxies

We put galaxies (their stars) into three groups:

1. Population I: metal-rich

2. Population II: metal-poor

3. Population III: metal-free

The metallicity is determined by the local conditions during stellar formation

The metal-poor stars are approximately 13Gyr old. Bulge stars are $(8-10)$Gyr. Some star forming regions in the Large Magellanic Cloud are only 10Myr old.