I. PATH INTEGRALS

In this section we consider imaginary-time Feynman path integrals, which is the first stepping stone in our development of the theory behind superconductivity, itinerant magnetism and non-equilibrium many-body physics. The idea of path integrals is to cut up a matrix element of the form $\langle f|\exp(-iHt)|i\rangle$ into small pieces and summing over all possible paths. Effectively

$$\langle f|\exp(-iHt)|i\rangle = \sum_{\text{paths}} \exp(iS_{\text{path}})$$

where we've defined the action as

$$S_{\text{path}} \equiv \int_0^t \mathrm{d}t' \ (p\dot{q} - H[p,q])$$

This is a precise formulation of quantum mechanics. This concept can readily be extended to imaginary times, such that we can treat temperature dependent problems. In the many-body problem the matrix element of interest is

$$Z = \mathrm{tr}\left[\exp(-\beta H)\right] = \sum_\lambda \langle\lambda|\exp(-iHt)|\lambda\rangle|_{t\to -i\beta}$$

where now, using imaginary times

$$Z = \sum_{\text{periodic paths}} \exp(-S_{\mathrm{E}}),$$

and

$$S_{\mathrm{E}} = \int_0^\beta \mathrm{d}\tau \ (-ip\partial_\tau q + H[p,q])$$

However, this idea can further be extended, such that we obtain a rich description of many-body physics, which we get by introducing *coherent states*. For a single boson field, a coherent state is given by

$$|b\rangle = e^{b\hat{\mathrm{b}}^\dagger}|0\rangle$$

similarly in the many-body case the coherent state is an eigenstate of the field operator, $\hat{\psi}$:

$$\hat{\psi}(x)|\phi\rangle = \phi(x)|\phi\rangle$$

This will allow us to effectively replace the fields $\hat{\psi}$ and $\hat{\psi}^\dagger$ with their eigenvalues ϕ and $\bar{\phi}$, however, this shouldn't blindly be done – for instance we must require that a Hamiltonian is normal ordered in order for us to perform the aforementioned replacement. Note that this also implies that time-ordered expectation values such as

$$\langle\mathbb{T}\ \hat{\psi}(1)\hat{\psi}^\dagger(2)\rangle = \frac{1}{Z}\sum_{\text{path}} \exp\left(-S_{\text{path}}\right)\phi(1)\bar{\phi}(2)$$

just become a functional integral over all possible paths connecting the points 1 and 2 (we no longer need to think about the operator properties of the fields).

A. Coherent States for Bosons

As defined above, the bosonic coherent state is

$$|b\rangle \equiv e^{b\hat{\mathrm{b}}^\dagger}|0\rangle = \sum_n \frac{b^n|n\rangle}{\sqrt{n!}}$$

where I have written out the exponential. This has a few properties that are noteworthy, for instance

$$\hat{\mathrm{b}}|b\rangle = b|b\rangle, \quad \langle b|\hat{\mathrm{b}}^\dagger = \langle b|\bar{b}$$

Note that the overlap between two coherent states is

$$\langle b_1|b_2\rangle = e^{\bar{b}_1 b_2}$$

Generally matrix elements of *normal ordered* operators, $\hat{O}[\hat{\mathrm{b}}^\dagger,\hat{\mathrm{b}}]$, can be evaluated by replacing the operators with their eigenvalues, and multiplying by the overlap:

$$\langle b_1|\hat{O}[\hat{\mathrm{b}}^\dagger,\hat{\mathrm{b}}]|b_2\rangle = O[\bar{b}_1,b_2]e^{\bar{b}_1 b_2}$$

The completeness relation must be modified slightly, due to the overlap between the states

$$\mathbb{1} = \int \frac{\mathrm{d}\bar{b}\mathrm{d}b}{2\pi i}e^{-\bar{b}b}|b\rangle\langle b|$$

We say that the coherent states form an *overcomplete* basis, which is similar to a complete basis, but the coefficients that describe an arbitrary state are not uniquely defined.

Bosonic Gaussian Integral

$$\int\prod_j \frac{\mathrm{d}\bar{b}_j\mathrm{d}b_j}{2\pi i}e^{-[\bar{\boldsymbol{b}}\cdot A\cdot\boldsymbol{b} - \bar{\boldsymbol{j}}\cdot\boldsymbol{b} - \bar{\boldsymbol{b}}\cdot\boldsymbol{j}]} = \frac{\exp\left[\bar{\boldsymbol{j}}\cdot A^{-1}\cdot\boldsymbol{j}\right]}{\det A}$$

B. Path Integral for the Partition Function: Bosons

We can make our $\sum_{\text{periodic paths}}$ a bit more rigorous by defining it as a functional integral over all possible b and $\bar{b}$, we obtain the working definition of the partition functional, which will be useful for the remainder of the course:

$$Z \equiv \int \mathcal{D}[\bar{b},b]\exp\left(-S[\bar{b},b]\right),$$

$$S[\bar{b},b] \equiv \int_0^\beta \mathrm{d}\tau \ \left(\bar{b}\partial_\tau b + H[\bar{b},b]\right)$$

This is obtained by slicing the interval $[0,\beta]$ up into infinitesimal slices and summing over all possible paths, we begin with:

$$\mathrm{tr}\left[\exp(-\beta H)\right] = \int \frac{\mathrm{d}\bar{b}\mathrm{d}b}{2\pi i} e^{-\bar{b}b} \langle b|\exp(-\beta H)|b\rangle$$

even if H is normal ordered, $\exp(H)$ is *not* normal ordered. So we will need to use that

$$\exp(-\beta H) = \lim_{N\to\infty} (1-\delta\tau H)^N$$

where $\delta\tau = \frac{\beta}{N}$, then we can insert a (over-)completeness relation inbetween each of the terms:

$$\int \prod_j \frac{\mathrm{d}\bar{b}_j \mathrm{d}b_j}{2\pi i} e^{-\bar{b}_j b_j} \langle b_j|\exp(-\delta\tau H)|b_{j-1}\rangle$$

where we've used that $b_0 = b_N$. Now the linearised exponential together with the normalisation factor gives us

$$\begin{aligned} e^{-\bar{b}_j b_j} \langle b_j|\exp(-\delta\tau H)|b_{j-1}\rangle &\approx e^{-\bar{b}_j b_j} \langle b_j|1-\delta\tau H|b_{j-1}\rangle \\ &e^{-\bar{b}_j b_j + \bar{b}_j b_{j-1}} \left(1-\delta\tau H[\bar{b}_j, b_{j-1}]\right) \\ &\approx \exp\left[-\delta\tau\left(b_j \frac{b_j - b_{j-1}}{\delta\tau} + H[\bar{b}_j, b_{j-1}]\right)\right] \end{aligned}$$

hence, in the continuum limit we obtain Equation I B. For multiple bosons we obtain a similar expression

> Bosonic Partition Function
>
> $$Z = \int \mathcal{D}[\bar{\boldsymbol{b}}, \boldsymbol{b}] e^{-S},$$
>
> $$S[\bar{\boldsymbol{b}}, \boldsymbol{b}] \equiv \int_0^\beta \left(\sum_\lambda \bar{b}_\lambda \partial_\tau b_\lambda + H[\bar{b}_\lambda, b_\lambda]\right)$$

where $\mathcal{D}[\bar{\boldsymbol{b}}, \boldsymbol{b}] = \prod_\lambda \mathcal{D}[\bar{b}_\lambda, b_\lambda]$. We can use these quantities to calculate time-ordered expectation values

$$G(2-1) = -\langle \mathbb{T}\, \hat{\mathrm{b}}(2)\hat{\mathrm{b}}^\dagger(1)\rangle = -\frac{\int \mathcal{D}[\bar{\boldsymbol{b}}, \boldsymbol{b}] e^{-S} b(2)\bar{b}(1)}{\int \mathcal{D}[\bar{\boldsymbol{b}}, \boldsymbol{b}] e^{-S}}$$

During this course we are particularly interested in *Gaussian Path Integrals*, where the action is a quadratic functional of the fields. For a free boson, with Hamiltonian $\hat{H} = \hat{\mathrm{b}}^\dagger_\alpha h_{\alpha\beta} \hat{\mathrm{b}}_\beta$ the partition function can be evaluated exactly:

$$Z_{\mathrm{G}} = \int \mathcal{D}[\bar{b}, b] \exp\left[-\int_0^\beta \mathrm{d}\tau \bar{b}(\partial_\tau + \underline{h})b\right] = \left[\det(\partial_\tau + \underline{h})\right]^{-1}$$

The matrix $\partial_\tau + \underline{h}$ is better thought of when it is expressed in Matsubara space, where $b(\tau) = \beta^{-\frac{1}{2}} \sum_n b(i\nu_n) e^{-i\nu_n \tau}$ and we get

$$(\partial_\tau + h_{\alpha\beta}) \to (-i\nu_n \delta_{\alpha\beta} + h_{\alpha\beta})$$

in which case we get

$$Z_{\mathrm{G}} = \frac{1}{\det\left[\partial_\tau + \underline{h}\right]} = \prod_{n,\lambda} \frac{1}{(-i\nu_n + \varepsilon_\lambda)}$$

where we have written the matrix out in Matsubara space, and decomposed the Hamiltonian to its eigenvalues, ε_λ. Note that this means

$$F_{\mathrm{G}} = -T \ln Z_{\mathrm{G}} = T \sum_{n,\lambda} \ln(\varepsilon_\lambda - i\nu_n) e^{i\nu_n 0^+}$$

which is the free energy of non-interacting bosons. Generally the matrix that stands in place of $(\partial_\tau + \underline{h})$ is closely related to the Green's matrix:

$$Z_{\mathrm{G}} = \int \mathcal{D}[\bar{b}, b] \exp\left[-\int_0^\beta \bar{b}(-G^{-1})b\right] = \left[\det\left(-G^{-1}\right)\right]^{-1}$$

so in general

$$F = T \ln \det\left[-G^{-1}\right] = T\, \mathrm{tr} \ln\left[-G^{-1}\right]$$

where the equality ln det = tr ln will be useful throughout the course. We can generalise our results such that they include source terms, so that we obtain the generating functional that was used throughout CMT1:

$$\begin{aligned} Z_{\mathrm{G}}[\bar{j}, j] &= \int \mathcal{D}[\bar{b}, b] e^{-\int_0^\beta \mathrm{d}1 \left[\bar{b}(\partial_\tau + \underline{h})b - \bar{j}(1)\cdot b(1) - \bar{b}(1)\cdot j(1)\right]} \\ &= \frac{e^{-\int_0^\beta \mathrm{d}1\mathrm{d}2\; \bar{j}(1) G(1-2) j(2)}}{\det[\partial_\tau + \underline{h}]} \end{aligned}$$

C. Fermions: Coherent States and Graßmann mathematics

The treatment for fermions follows the bosonic analysis quite closely, except that there is a fundamental difference in how we treat the eigenvalues of the field operators. For bosons we replaced $\hat{\psi} \to \phi$ and treated ϕ as a number. However, due to the fact that fermionic operators *anti-commute* we need their eigenvalues to commute too, which inspires us to use Graßmann numbers.

Similarly to before we define a coherent state as

$$|c\rangle = \exp\left(\hat{\mathrm{c}}^\dagger c\right)|0\rangle, \qquad \langle c| = \langle 0| \exp(\bar{c}\hat{\mathrm{c}})$$

where the numbers $c, \bar{c} \in \mathcal{G}$ *anti-commute* with themselves and with the field operators:

$$c\bar{c} + \bar{c}c = 0, \quad c\hat{\psi} + \hat{\psi}c = 0$$

this also implies that

$$c^2 = 0, \quad \bar{c}^2 = 0$$

(due to the fact that $(\hat{\psi}^{(\dagger)})^2 |\Psi\rangle = |0\rangle$). This tells us that Taylor expansion of functions are almost trivial:

$$f(\bar{c}, c) = f_0 + \bar{c} f_1 + \bar{f}_1 c + f_2 \bar{c} c$$

where $f_0, f_1, \bar{f}_1, f_2 \in \mathbb{C}$. This is the full expansion; higher order terms are zero. From here we are ready to use the results from the bosonic case,

$$\sum_{\bar{b}b} = \int \frac{\mathrm{d}\bar{b}\mathrm{d}b}{2\pi i} e^{-\bar{b}b} \to \sum_{\bar{c}c} = \int \mathrm{d}\bar{c}\mathrm{d}c e^{-\bar{c}c}$$

note the lack of the $2\pi i$ in the denominator for fermions. Trace formulae contain an extra minus:

$$\mathrm{tr}\, A = \sum_{\bar{c}c} \langle -\bar{c}|A|c\rangle$$

and Jacobians as well as Gaussian integrals are the inverses of their bosonic counterparts:

Fermionic Gaussian Integral

$$\int \prod_j \mathrm{d}\bar{c}_j \mathrm{d}c_j e^{-[\bar{\boldsymbol{c}}\cdot A\cdot \boldsymbol{c} - \bar{\boldsymbol{j}}\cdot \boldsymbol{c} - \bar{\boldsymbol{c}}\cdot \boldsymbol{j}]} = \det A \exp\left[\bar{\boldsymbol{j}} \cdot A^{-1} \cdot \boldsymbol{j}\right]$$

notice how it is no longer the inverse of the determinant that appears. The completeness relation is rather simple

$$\int \mathrm{d}\bar{c}\mathrm{d}c |c\rangle\langle\bar{c}| e^{-\bar{c}c} = |0\rangle\langle 0| + |1\rangle\langle 1| \equiv \mathbb{1}$$

The extension from one fermion to many fermions is as simple it was for bosons, but keep in mind that all Graßmann numbers anti-commute with each other.

D. Path Integral for the Partition Function: Fermions

We once again begin with

$$Z = -\int \mathrm{d}\bar{c}_N \mathrm{d}c_0 \; e^{\bar{c}_N c_0} \langle \bar{c}_N | \exp(-\beta H) | c_0 \rangle$$

and use that $\exp(-\beta H) = \lim_{N\to\infty}(1 - \delta\tau H)^N$, so that

$$Z = -\int \mathrm{d}c_N \mathrm{d}c_0 \; e^{\bar{c}_n c_0} \prod_{j=1}^{N-1} \mathrm{d}\bar{c}_j \mathrm{d}c_j e^{-\bar{c}_j c_j} \times \prod_{j=1}^{N} \langle \bar{c}_j | \exp(-\delta\tau H) | c_{j-1} \rangle$$

previously we used that $b_N = b_0$, but for fermions $c_N = -c_0$. Following similar steps as we did for the bosonic case we obtain

$$S = \int_0^\infty \mathrm{d}\tau \left[\bar{c}(\partial_\tau + \varepsilon)c\right], \quad Z = \int \mathcal{D}[\bar{c}, c] \exp\left[-S\right]$$

where $\mathcal{D}[\bar{c}, c] = \prod_j \mathrm{d}\bar{c}_j \mathrm{d}c_j$. We can transform to Matsubara space, where we now use fermionic frequencies, ω_n. Obtaining

$$S = \sum_n \bar{c}_n(-i\omega_n + \varepsilon)c_n$$

E. The Hubbard-Stratonovich Transformation

The Hubbard-Stratonovich transformation is a transformation in the action, where we introduce new fields such that we can complete the square and simplify the expression to obtain a palatable effective action.

Consider a fermionic path integral of the form

$$Z_0 = \int \mathcal{D}[\bar{\boldsymbol{c}}, \boldsymbol{c}] \exp\left[-\int_0^\beta \mathrm{d}\tau \left(\bar{\boldsymbol{c}}(\partial_\tau + \underline{h})\boldsymbol{c} - g\bar{\boldsymbol{A}} \cdot \boldsymbol{A}\right)\right]$$

where $\boldsymbol{A}$ is an electron bilinear. Now let us multiply this by a *number*

$$Z_\alpha = \int \mathcal{D}[\bar{\boldsymbol{\alpha}}, \boldsymbol{\alpha}] \exp\left[-\sum_j \int_0^\beta g^{-1} \bar{\boldsymbol{\alpha}} \cdot \boldsymbol{\alpha}\right]$$

This is just a Gaussian integral which we can evaluate exactly. Let us consider $Z = Z_0 \times Z_\alpha$, giving an effective action:

$$S = \int_0^\beta \mathrm{d}\tau \left[\bar{\boldsymbol{c}}\left(-\underline{G}_0^{-1}\right)\boldsymbol{c} - g\bar{\boldsymbol{A}} \cdot \boldsymbol{A} - g^{-1}\bar{\boldsymbol{\alpha}} \cdot \boldsymbol{\alpha}\right]$$

where $S = S_0 + S_\alpha$. The partition function is given by a functional integral of the action above, so we can shift the variable α freely, just like one would do for regular integrals. If we choose to shift it such that the new integration “variable"is

$$\boldsymbol{\Delta} = \boldsymbol{\alpha} - g\boldsymbol{A}$$

we get that

$$-g\bar{\boldsymbol{A}}\cdot\boldsymbol{A} - g^{-1}\bar{\boldsymbol{\alpha}}\cdot\boldsymbol{\alpha} = \bar{\boldsymbol{A}}\cdot\boldsymbol{\Delta} + \bar{\boldsymbol{\Delta}}\cdot\boldsymbol{A} + g^{-1}\bar{\boldsymbol{\Delta}}\cdot\boldsymbol{\Delta}$$

hence we obtain

$$Z = \int \mathcal{D}[\boldsymbol{\Delta},\boldsymbol{c}]\exp\left[-S[\boldsymbol{\Delta},\boldsymbol{c}]\right]$$

$$S[\boldsymbol{\Delta},\boldsymbol{c}] = \int_0^\beta \mathrm{d}\tau\; g^{-1}\bar{\boldsymbol{\Delta}}\cdot\boldsymbol{\Delta} + \bar{\boldsymbol{c}}\left(-\underline{G}_0^{-1}\right)\boldsymbol{c} + \bar{\boldsymbol{A}}\cdot\boldsymbol{\Delta} + \bar{\boldsymbol{\Delta}}\cdot\boldsymbol{A}$$

Notice how all terms either don't depend on $\boldsymbol{c}$ or are bilinears, which implies that this is an action that can be integrated by means of our Gaussian integration technique described above. However, now once the electrons are integrated out* we obtain an effective action in the $\boldsymbol{\Delta}$ fields.

$$S_{\mathrm{E}}[\bar{\boldsymbol{\Delta}},\boldsymbol{\Delta}] = \int \mathrm{d}\tau g^{-1}\bar{\boldsymbol{\Delta}}\cdot\boldsymbol{\Delta} - \mathrm{tr}\,\ln[\partial_\tau + \underline{h}_{\mathrm{E}}[\bar{\boldsymbol{\Delta}},\boldsymbol{\Delta}]]$$

1. *Coulomb Force*

In the case of the three-dimensional Coulomb interaction the interacting-Hamiltonian in Fourier space is

$$H_{\mathrm{I}} = \frac{1}{2}\int_{\boldsymbol{q}} \frac{e^2\hat{\rho}_{\boldsymbol{q}}\hat{\rho}_{-\boldsymbol{q}}}{\varepsilon_0 q^2}$$

where $\hat{\rho}_{\boldsymbol{q}} = \hat{c}^\dagger_{\boldsymbol{q}}\hat{c}_{\boldsymbol{q}}$. We see that this part of the Hamiltonian is quartic in the fields, so it is suitable to be integrated away by means of the Hubbard-Stratonovich transformation. Effectively we can add a bilinear in a new field $\phi_{\boldsymbol{q}}$ to the Hamiltonian and then perform the change of variables. This corresponds exactly to what was done above when we multiplied the partition function by a Gaussian integral. Let us choose

$$H_{\mathrm{I}} \to H_{\mathrm{I}}' = \frac{1}{2}\int_{\boldsymbol{q}}\left[\frac{e^2\rho_{\boldsymbol{q}}\rho_{-\boldsymbol{q}}}{\varepsilon_0 q^2} - \varepsilon_0 q^2\phi_{\boldsymbol{q}}\phi_{-\boldsymbol{q}}\right]$$

and now we let

$$\phi_{\boldsymbol{q}} \to \phi_{\boldsymbol{q}} - \frac{e\rho_{\boldsymbol{q}}}{\varepsilon_0 q^2}$$

to obtain

$$H_{\mathrm{I}}' = \int_{\boldsymbol{q}}\left(e\rho_{\boldsymbol{q}}\phi_{-\boldsymbol{q}} - \frac{\varepsilon_0}{2}q^2\phi_{\boldsymbol{q}}\phi_{-\boldsymbol{q}}\right)$$

which is a bilinear in both fields and hence can be handled more easily.

* The process of performing the $\boldsymbol{c}$ integration is referred to as integrating the electrons out.

II. PATH INTEGRALS AND ITINERANT MAGNETISM

One application of the Hubbard-Stratonovich transformation is in describing itinerant magnetism. The Weiss field, $\boldsymbol{M}$, which is more commonly known as the magnetisation, is a quantity one normally encounters in a mean-field-theory discussion of magnetism. However, this field appears in the path integral when we perform a Hubbard-Stratonovich transformation, as we will see shortly. Let us begin with the Hubbard model

$$H = \sum_{\boldsymbol{k}\sigma}\varepsilon_{\boldsymbol{k}} c^\dagger_{\boldsymbol{k}\sigma}c_{\boldsymbol{k}\sigma} + U\sum_j n_{j\uparrow}n_{j\downarrow}$$

which describes electrons on a tight-binding lattice. Notice how the electron-electron interaction can be written as

$$Un_{j\uparrow}n_{j\downarrow} = \frac{U}{2}\left[(n_{j\uparrow}+n_{j\downarrow}) - (n_{j\uparrow}-n_{j\downarrow})^2\right]$$

the first term is just a shift in the chemical potential so we will omit it, however the second term is related to the spin along the z-axis. Due to the isotropy of the system we can express this as a third of the average spin squared in all three directions:

$$(n_{j\uparrow}-n_{j\downarrow})^2 = \frac{1}{3}|\boldsymbol{S}_j|^2, \qquad \boldsymbol{S}_j \equiv \sum_{\alpha\beta} c^\dagger_{j\alpha}\boldsymbol{\sigma}_{\alpha\beta}c_{j\beta}$$

so

$$Un_{j\uparrow}n_{j\downarrow} \to I|\boldsymbol{S}_j|^2, \qquad 3I = U$$

So our action is

$$S = \int_0^\beta \mathrm{d}\tau\left[\sum_{\boldsymbol{k}\sigma}\bar{c}_{\boldsymbol{k}\sigma}(\partial_\tau + \varepsilon_k)c_{\boldsymbol{k}\sigma} - \frac{I}{2}\sum_j|\boldsymbol{S}_j|^2\right]$$

now we introduce the Hubbard-Stratonovich field

$$-\frac{I}{2}\sum_j|\boldsymbol{S}_j|^2 \to -\frac{I}{2}\sum_j|\boldsymbol{S}_j|^2 + \sum_j\frac{|\boldsymbol{m}_j|^2}{2I}$$

and shift $\boldsymbol{m}$ such that it absorbs the quartic term

$$\boldsymbol{m}_j = \boldsymbol{M}_j - I\boldsymbol{S}_j$$

leaving

$$S=\int_0^\beta \mathrm{d}\tau\left[\sum_{\boldsymbol{k}\sigma}\bar{c}_{\boldsymbol{k}\sigma}(\partial_\tau + \varepsilon_k)c_{\boldsymbol{k}\sigma} - \sum_j\left[\boldsymbol{M}_j\cdot\boldsymbol{S}_j - \frac{|\boldsymbol{M}_j|^2}{2I}\right]\right]$$

which is a bilinear in the electron fields, so we are ready to integrate out the electrons:

$$Z = \int\mathcal{D}[\boldsymbol{M}]e^{-S_{\mathrm{E}}[\boldsymbol{M}]}, \quad e^{-S_{\mathrm{E}}[\boldsymbol{M}]} = \int\mathcal{D}[\bar{c},c]e^{-S[\bar{c},c,\boldsymbol{M}]}$$

where once we integrate out the electrons

$$e^{-S_{\mathrm{E}}[\boldsymbol{M}]} = \det\left[\partial_\tau + h_{\mathrm{E}}[\boldsymbol{M}]\right] \exp\left[-\sum_j \int_0^\beta \mathrm{d}\tau\, \frac{|\boldsymbol{M}_j|^2}{2I}\right]$$
$$h_{\mathrm{E}}[\boldsymbol{M}] \equiv \varepsilon_{\boldsymbol{k}}\delta_{\boldsymbol{k},\boldsymbol{k}'} - \boldsymbol{M}_{\boldsymbol{k}-\boldsymbol{k}'}\cdot\boldsymbol{\sigma}$$

If we write $G_0 = (i\omega - \varepsilon_{\boldsymbol{k}})^{-1}$ and $V_{k,k'} = \boldsymbol{M}_{k-k'}\cdot\boldsymbol{\sigma}$ then the effective action can be expressed as

$$S_{\mathrm{E}}[\boldsymbol{M}] = -\operatorname{tr}\ln\left[-G_0^{-1}(k)\right] - \operatorname{tr}\ln\left[1 - G_0(k)V_{k,k'}\right] + N\beta\sum_q \frac{|\boldsymbol{M}_q|^2}{2I}$$

where k and q are Matsubara-four-momenta. The second term can be expanded

$$\operatorname{tr}\ln(1 - G_0V) = \operatorname{tr}\left[-G_0V - \frac{1}{2}(G_0V)^2 - \frac{1}{3}(G_0V)^3 + \cdots\right]$$

which can be written in terms of Feynman diagrams where an electron scatters off the Weiss field n times (for the n-th term). Hence we obtain an expression for the free energy in terms of an infinite sum of Feynman Diagrams, just like we did in CMT1, however, this time by means of a very different method.

A. Saddle Points and the Mean-Field Theory of the Magnetism

In the saddle-point approximation we assume that the partition is equal to its value at the saddle point, for instance where $\boldsymbol{M} = \boldsymbol{M}_0$, where

$$\left.\frac{\delta S_{\mathrm{E}}[\boldsymbol{M}]}{\delta \boldsymbol{M}}\right|_{\boldsymbol{M}=\boldsymbol{M}_0} = 0$$

we assume that

$$Z = \int \mathcal{D}[\boldsymbol{M}]\exp\left[-S_{\mathrm{E}}[\boldsymbol{M}]\right] \approx \exp\left[-S_{\mathrm{E}}[\boldsymbol{M}_0]\right]$$

let us evaluate Equation II A using Equation II:

$$\frac{\delta S_{\mathrm{E}}}{\delta \boldsymbol{M}_j} = e^{S_{\mathrm{E}}}\int \mathcal{D}[\bar{c},c]\left(\frac{\boldsymbol{M}_j}{I} - \bar{c}_j\boldsymbol{\sigma}c_j\right)e^{-S[\bar{c},c,\boldsymbol{M}]}$$

requiring that this is zero means

$$\boldsymbol{M}_j^0 = I\langle c_j^\dagger\boldsymbol{\sigma}c_j\rangle$$

which is exactly what the mean-field-theory approach would have begun with. We're often interested in a static saddle plot, such that $\boldsymbol{M}_j^0$ doesn't depend on τ, in which case

$$e^{-S_{\mathrm{E}}[\boldsymbol{M}^0]} = \operatorname{tr}\left[e^{-\beta\hat{H}_{\mathrm{MF}}}\right], \quad \hat{H}_{\mathrm{MF}} = c^\dagger \underline{h}_E[\boldsymbol{M}^0]c + \sum_j \frac{|\boldsymbol{M}_j^0|^2}{2I}$$

Simplifying things further by assuming that we are describing a ferromagnet, which has a uniform magnetisation $\boldsymbol{M}_j^0 = M\hat{\boldsymbol{z}}$, in which case we get

$$H_{\mathrm{MF}} = \sum_{\boldsymbol{k}\sigma} c_{\boldsymbol{k}\sigma}^\dagger(\varepsilon_{\boldsymbol{k}} - \sigma_z M)c_{\boldsymbol{k}\sigma} + N\frac{M^2}{2I}$$

which implies that

$$S_{\mathrm{E}} = -\sum_{\boldsymbol{k},i\omega_n}\operatorname{tr}\ln\left[\varepsilon_{\boldsymbol{k}} - M\sigma_z - i\omega_n\right] + N\beta\frac{M^2}{2I}$$

from which we can obtain information about the macroscopic system. Performing the Matsubara sum:

$$F_{\mathrm{E}}[M] = -\frac{1}{N\beta}\sum_{\boldsymbol{k},\sigma=\pm1}\ln\left[1 + e^{-\beta(\varepsilon_{\boldsymbol{k}} - \sigma M)}\right] + \frac{M^2}{2I}$$

For instance by evaluating the free energy and minimising it we can obtain Stoner's criterion for ferromagnetic order.

Let us use this result to derive the Landau expansion of the free energy. We have†

$$-\frac{T}{2}\sum_{\sigma=\pm1}\ln\left[1 + e^{-\beta(\varepsilon - \sigma M)}\right]$$
$$= -T\ln\left[1 + e^{-\beta\varepsilon}\right] + \sum_{r=1}^\infty \frac{M^{2r}}{(2r)!}\frac{\mathrm{d}[}{\mathrm{d}2}r - 1]f(\varepsilon)\varepsilon$$

So

$$F[M] = F_0 + \sum_{r=1}^\infty \frac{M^{2r}}{(2r)!}\int \mathrm{d}\varepsilon\, N(\varepsilon)\frac{\mathrm{d}[}{\mathrm{d}2}r - 1]f(\varepsilon)\varepsilon + \frac{M^2}{2I}$$

integrating by parts so that we only have one derivative of the Fermi-Dirac function we obtain, up to fourth order in M:

$$F = F_0 + \frac{1}{2}M^2\left(I^{-1} - \chi_0(T)\right) + \frac{M^4}{4!}\overline{(-N''(0))} + \mathcal{O}(M^6)$$

† odd terms die due to the σ sum.

where

$$\chi_0(T) \equiv -\int \mathrm{d}\varepsilon \; N(\varepsilon) \frac{\partial f(\varepsilon)}{\partial \varepsilon}$$

and $\overline{N''(0)}$ denotes the thermal average of the second derivative of the density of states around the Fermi energy.

B. Quantum Fluctuations in the Magnetisation

Let us now consider a system that is *close* to the saddle point, that is

$$\boldsymbol{M}_j(\tau) = \boldsymbol{M}^0 + \delta\boldsymbol{M}_j(\tau) \leftrightsquigarrow \boldsymbol{M}_q = \boldsymbol{M}^0 \delta_{q=0} + \delta M_q$$

where $q = (\boldsymbol{q}, i\nu_n)$. Near the saddle point

$$S_{\mathrm{E}}[\boldsymbol{M}] = S_{\mathrm{E}}[\boldsymbol{M}^0] + \frac{1}{2} \sum_{q,m,n} \frac{\delta^2 S}{\delta M_q^m \delta M_{-q}^n} \delta \boldsymbol{M}_q^m \delta \boldsymbol{M}_{-q}^n + \mathcal{O}(\delta \boldsymbol{M}^3)$$

The linear term disappears because we are at a saddle point. Provided the fluctuations are small compared to the order parametre we can use this quadratic approximation to describe magnetic fluctuations.

The renormalised Green's function before we've taken the magnetisation fluctuations into account is

$$= \underline{G}_k = (i\omega_n - \varepsilon_{\boldsymbol{k}} - \sigma_z M)^{-1}$$

Now if we expand the effective action by substituting $\boldsymbol{M}_{k-k'} = \boldsymbol{M}\delta_{k-k'} + \delta\boldsymbol{M}_{k-k'}$:

$$F_{\mathrm{E}}[\boldsymbol{M}] = -\frac{1}{N\beta} \operatorname{tr} \ln\left(-\underline{G}_k^{-1} \delta_{k-k'} - \delta \boldsymbol{M}_{k-k'} \cdot \boldsymbol{\sigma}\right) + \sum_q \frac{\left|M\hat{\boldsymbol{z}}\delta_q + \delta\boldsymbol{M}_q\right|^2}{2I}$$

We can write this out as a sum of Feynman Diagrams, just like in Equation II. However, only looking at the *change* in the free energy will remove one of the terms, which leaves us with

$$\Delta F_{\mathrm{E}}[\boldsymbol{M}] = -\left[\qquad + \qquad + \cdots \right] + \sum_q \frac{|\delta \boldsymbol{M}_q|^2}{2I}$$

so if we just include the quadratic terms (remember each vertex carries weight $\sim \delta\boldsymbol{M}$) we obtain

$$\Delta F_{\mathrm{G}}[\boldsymbol{M}] = \frac{1}{2} \sum_q \delta \boldsymbol{M}_q \left[\frac{\mathbb{1}}{I} - \underline{\chi}_q^{(0)} \right] \delta \boldsymbol{M}_{-q}$$

where the bare susceptibility of the polarised metal is defined by

$$\underline{\chi}_{mn}^{(0)}(q) = -\frac{1}{\beta N} \sum_k \operatorname{tr}\left[\sigma_m \underline{G}_{k+q} \sigma_n \underline{G}_k\right]$$

the presence of the σ_z term in the Green's function means that χ contains off diagonal components (if $\underline{G} \sim \mathbb{1}$ the we get a δ_{mn}). It is convenient to change basis, so that we are working in the raising and lowering basis of the transverse spin:

$$M_q^{\pm} = M_q^x \pm i M_q^y, \quad \sigma_{\pm} = \frac{1}{2}(\sigma_x \pm i \sigma_y)$$

where now the nonzero components of the transverse susceptibility are

$$\chi_{\pm\mp}^{(0)}(q) = -\frac{1}{\beta N} \sum_k \operatorname{tr}\left[\sigma_{\pm} \underline{G}_{k+q} \sigma_{\mp} \underline{G}_q\right]$$

notive how $\chi_{-+}(q) = \chi_{+-}(-q)$ which can be shown by shifting the k-sum by $k \to k+q$ and using the cyclicity of the trace. Now writing $\delta\boldsymbol{M} \cdot \boldsymbol{\sigma} = \delta M^z \sigma_z + \delta M^+ \sigma_- + \delta M^- \sigma_+$, and using that the magnetisation is a real variable implying that $\delta \bar{M}_q^{\pm} = \delta M_{-q}^{\mp}$ we obtain

$$\begin{aligned}\Delta F_{\mathrm{G}}[\boldsymbol{M}] =& \frac{1}{2} \sum_q \Big[\delta M_{-q}^z \left(I^{-1}\mathbb{1} + \chi_{zz}^{(0)} \right) \delta M_q^z \\ &+ \delta \bar{M}_q^+ \left((2I)^{-1}\mathbb{1} + \chi_{+-}^{(0)}(q) \right) \delta M_q^+ \\ &+ \delta \bar{M}_{-q}^+ \left((2I)^{-1}\mathbb{1} + \chi_{-+}^{(0)}(q) \right) \delta M_{-q}^+ \Big]\end{aligned}$$

which simplifies to

$$\begin{aligned}\Delta F_{\mathrm{G}}[\boldsymbol{M}] = \sum_q \Big[&\frac{1}{2} \delta M_{-q}^z \left(I^{-1}\mathbb{1} + \chi_{zz}^{(0)} \right) \delta M_q^z \\ &+ \delta \bar{M}_q^+ \left((2I)^{-1}\mathbb{1} + \chi_{+-}^{(0)}(q) \right) \delta M_q^+ \Big]\end{aligned}$$

when we use $\delta \bar{M}_q^{\pm} = \delta M_{-q}^{\mp}$. So we have obtained a distribution function for the Gaussian magnetic fluctuations about the Stoner mean-field theory for an itinerant ferromagnet:

$$p[\boldsymbol{M}] \sim e^{-\Delta S[\boldsymbol{M}]} = e^{-\beta N \Delta F_{\mathrm{G}}[\boldsymbol{M}]}$$

From this Gaussian form we can directly read off the fluctuations in the magnetisation

$$\langle \delta M_q^{\alpha} \delta M_{-q'}^{\beta} \rangle = \frac{1}{\beta N} \delta_{qq'} \langle \delta M_q^{\alpha} \delta M_{-q}^{\delta} \rangle$$

where

$$\langle \delta M_q^z \delta M_{-q}^z \rangle = \frac{1}{I^{-1} - \chi_{zz}^{(0)}(q)}$$

$$\langle \delta M_q^+ \delta \bar{M}_{-q}^+ \rangle = \frac{1}{(2I)^{-1} - \chi_{+-}^{(0)}(q)}$$

which we can relate to spin-correlation functions by referring back to the Hubbard-Stratonovich transformation we performed (Equation II):

$$IS_j^\alpha = M_j^\alpha - m_j^\alpha \;\rightsquigarrow\; I^2\langle S_j^\alpha S_j^\beta\rangle = \langle M_j^\alpha M_j^\beta\rangle + \langle m_j^\alpha m_j^\beta\rangle$$

the cross-terms $\langle M^\alpha m^\beta\rangle = 0$ because m and M are independent variables‡. We have that $\langle m_q^\alpha m_{-q}^\beta\rangle = I\delta_{\alpha\beta}$ and hence

Longitudinal	Transerval
$\langle \sigma_q^z \sigma_{-q}^q\rangle = \dfrac{\chi_{zz}^{(0)}(q)}{1 - I\chi_{zz}^{(0)}(q)}$	$\langle \sigma_q^z \sigma_{-q}^q\rangle = \dfrac{\chi_{+-}^{(0)}(q)}{1 - I\chi_{+-}^{(0)}(q)}$

these are the *RPA spin fluctuations* of itinerant ferromagnetism.

C. Linhard Function

We can evaluate the $\chi_{+-}(q)$ exactly in the case that $\varepsilon_k = \frac{k^2}{2m} - \mu$.

$$\begin{aligned}
\chi_{+-}^{(0)}(q) &= -\frac{1}{N\beta}\sum_{\boldsymbol{k},i\omega_n} \mathrm{tr}\left[\sigma_+ \underline{G}_{k+q}\sigma_- \underline{G}_k\right] \\
&= -\frac{1}{N\beta}\sum_{\boldsymbol{k},i\omega_n} \mathrm{tr}\left[\underline{G}_\downarrow(k+q)\underline{G}_\uparrow(k)\right] \\
&= a^3\int_{\boldsymbol{k}} \frac{f_{\boldsymbol{k}\uparrow} - f_{\boldsymbol{k}+\boldsymbol{q}\downarrow}}{(\varepsilon_{\boldsymbol{k}+\boldsymbol{q}\downarrow} - \varepsilon_{\boldsymbol{k}\uparrow}) - i\nu_n}
\end{aligned}$$

where $\varepsilon_{\boldsymbol{k}\sigma} = \varepsilon_{\boldsymbol{k}} - \mathrm{sgn}(\sigma)M$, which we can evaluate exactly by expanding this into the sum of to terms and changing variables such that only $f_{\boldsymbol{k}\sigma}$ appears. We get

$$\chi_{+-}^{(0)}(\boldsymbol{q},\nu) = \frac{1}{2}\sum_\sigma \left(\frac{mk_{F\sigma}}{\pi^2}\right)\mathcal{F}\left(\frac{q}{2k_{F\sigma}}, \sigma\left(\frac{\nu - 2m}{4\varepsilon_F}\right)\right)$$

where§ $k_{F\sigma} = k_F\left(1 + \mathrm{sgn}(\sigma)M\varepsilon_F^{-1}\right)^{\frac{1}{2}}$ and the Lindhard function is

$$\mathcal{F}(x,y) = \frac{1}{8x}\left[\left(1 - A^2\right)\ln\left(\frac{A+1}{A-1}\right) + 2A\right], \quad A \equiv x - x^{-1}y$$

‡ When one performs a Legendre transformation to go from a Lagrangian to a Hamiltonian the variables one introduces are also independent of initial variables

§ Note that Coleman doesn't have the σ in his definition of $k_{F\sigma}$

III. SUPERCONDUCTIVITY AND BCS THEORY

In this section we will discuss BCS Theory and how it relates to Ginzburg-Landau theory. But first we need to show that phonon-mediation can lead to an *attractive* electron-electron interaction

A. Bardeen-Pines

When describing superconductors it is important to remember that it is not only the electrons interacting with each other, but there is also a positive ionic background that contributes. The ionic background is more inert than the electrons – the proton to electron mass ratio is $\sim$ 1000, which means that the positive ions move far slower than electrons do. Define the ionic plasma frequency as

$$\Omega_{\mathrm{P}}^2 = \frac{Ze^2 n}{\varepsilon_0 M}$$

which is thousands of times smaller than the electronic plasma frequency. The ionic plasma frequency sets the length scale of charge fluctuations in the ionic background.

The polarisability not only contains an electronic contribution, χ_0, but also an ionic contribution χ_{ion}, so the RPA effective interaction is of the form

$$V_{\mathrm{eff}} = \frac{1}{N}\frac{V(q)}{1 + V(q)\left[\chi_0(q) + \chi_{\mathrm{ion}}(q)\right]} = \frac{V(q)}{\varepsilon(q)}$$

where $\chi_0 + \chi_{\mathrm{ion}}$ are the electronic and ionic polarisability bubbles. In the low-frequency, large $\frac{M}{m}$ limit we have

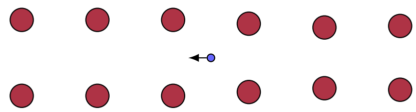
$$\varepsilon(q) = 1 + \frac{\kappa^2}{q^2} - \frac{\Omega_{\mathrm{P}}^2}{\nu^2}$$

which means

$$V_{\mathrm{eff}}(\boldsymbol{q},\nu) = \frac{e^2}{\varepsilon_0(q^2+\kappa^2)}\left(1 + \frac{\omega_q^2}{\nu^2 - \omega_q^2}\right), \quad \omega_q^2 = \frac{\Omega_{\mathrm{P}}^2 q^2}{q^2+\kappa^2}$$

the first term is the effective potential due to the electrons, which is frequency independent. The second term on the other hand is the frequency-dependent retarded electron-phonon interaction, which under the right circumstances can become negative – resulting in an attractive effective electron-electron interaction.

A more physical picture is given below



Figur 1: *The ions behind the electron have a slightly higher density because they have been pulled inwards, thus other electrons are attracted to a point behind the original electron (which is why it's considered retarded)*

B. The Cooper Instability

Leon Cooper showed that an arbitrarily small attractive force between electrons just above the Fermi sea gives rise to a two-particle bound state and a destabilisation of the Fermi-surface. Cooper imagined a state

$$|\psi(\boldsymbol{p})\rangle = \Lambda^\dagger(\boldsymbol{p})\,|FS\rangle\,,$$
$$\Lambda^\dagger(\boldsymbol{p}) \equiv \int \mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{x'}\;\phi_{\boldsymbol{x}-\boldsymbol{x'}}\psi^\dagger_{\downarrow\boldsymbol{x}}\psi^\dagger_{\uparrow\boldsymbol{x'}}e^{-i\boldsymbol{p}\cdot(\boldsymbol{x}+\boldsymbol{x'})/2}$$

$\Lambda^\dagger$ is the Cooper pair creation operator, which we can rewrite in terms of the momentum-dependent field operators

$$\Lambda^\dagger_{\boldsymbol{p}} = \sum_{\boldsymbol{k}} \phi_{\boldsymbol{k}} c^\dagger_{\boldsymbol{k}+\boldsymbol{p}/2\uparrow} c^\dagger_{-\boldsymbol{k}+\boldsymbol{p}/2\downarrow}$$

Instead of expressing $|\psi(\boldsymbol{p})\rangle$ in terms of the Fermi-surface we can define a state $|\boldsymbol{k},\boldsymbol{p}\rangle \equiv c^\dagger_{\boldsymbol{k}+\boldsymbol{p}/2\uparrow} c^\dagger_{-\boldsymbol{k}+\boldsymbol{p}/2\downarrow}|FS\rangle$ in which case we simply have

$$|\psi(\boldsymbol{p})\rangle = \sum_{\boldsymbol{k}} \phi_{\boldsymbol{k}}\,|\boldsymbol{k},\boldsymbol{p}\rangle$$

If we now consider a Hamiltonian of the form

$$H = \sum_{\boldsymbol{k}} \varepsilon_{\boldsymbol{k}} c^\dagger_{\boldsymbol{k}\sigma} c_{\boldsymbol{k}\sigma} + V$$

where V contains the details about the electron-electron interactions. If we assume that $|\psi(\boldsymbol{p})\rangle$ is an eigenstate that $H\,|\psi(\boldsymbol{p})\rangle = E_{\boldsymbol{p}}\,|\psi(\boldsymbol{p})\rangle$ and we obtain

$$E_{\boldsymbol{p}}\sum_{\boldsymbol{k}}\phi_{\boldsymbol{k}}\,|\boldsymbol{kp}\rangle = \sum_{|\boldsymbol{k}\pm\boldsymbol{p}/2|>k_F} (\varepsilon_+ - \varepsilon_-)\,\phi_{\boldsymbol{k}}\,|\boldsymbol{kp}\rangle + \sum_{|\boldsymbol{k}|,|\boldsymbol{k'}|>k_F} |\boldsymbol{kp}\rangle\,\langle \boldsymbol{kp}|V|\boldsymbol{k'p}\rangle\,\phi_{\boldsymbol{k'}}$$

where $\varepsilon_\pm = \varepsilon_{\boldsymbol{k}\pm\boldsymbol{p}/2}$. Now assume that $\langle \boldsymbol{kp}|V|\boldsymbol{k'p}\rangle\,\phi_{\boldsymbol{k'}} = -g_0/\mathcal{V}$, then

$$E_{\boldsymbol{p}}\phi_{\boldsymbol{k}} = (\varepsilon_+ - \varepsilon_-)\phi_{\boldsymbol{k}} - \frac{g_0}{\mathcal{V}}\sum_{0<\varepsilon_\pm<\omega_D}\phi_{\boldsymbol{k'}}$$

where ω_D is the cutoff energy above which the phonon mediation of the electron-electron interaction no longer makes the effective interaction attractive. More accurately, we say that Cooper pairs can only form within the interval $-\omega_D \le \varepsilon \le \omega_D$, so both above and below the Fermi-energy. We can rearrange and obtain

$$1 - \frac{g_0}{\mathcal{V}}\sum_{0<\varepsilon_\pm<\omega_D}\frac{1}{\varepsilon_+ - \varepsilon_- - E_{\boldsymbol{p}}} = 0$$

It is now convenient to define a function $\mathcal{G}^{-1}(E_{\boldsymbol{p}},\boldsymbol{p})$ whose zeros we would like to find:

$$\mathcal{G}^{-1}(z,\boldsymbol{p}) = 1 - g_0\chi_0(z,\boldsymbol{p}),$$
$$\chi_0(z,\boldsymbol{p}) = \frac{1}{\mathcal{V}}\sum_{0<\varepsilon_\pm<\omega_D}\frac{1}{\varepsilon_+ - \varepsilon_- - z}$$

for a quadratic dispersion we can solve this to obtain

$$\mathcal{G}^{-1}(E,p) = 1 - \frac{g_0 N(0)}{2}\ln\left(\frac{2\omega_D}{v_F p - E}\right)$$

thus we satisfy the Schrödinger equation by setting this equal to zero:

$$E_{\boldsymbol{p}} = -2\omega_D\exp\left[-\frac{2}{g_0 N(0)}\right] + v_F p$$

the linear spectrum is a signature of a collective bosonic mode. Notice how this energy is negative, which implies that it is energetically *beneficial* to form Cooper Pairs – hence the name "Cooper Instability".

C. The BCS Hamiltonian

However, this Cooper-pair state that we discussed in the previous section is not sufficient to understand superconductivity. We need to create a coherent state with the Cooper pair operator:

$$|\psi_{\mathrm{BCS}}\rangle = \exp\left(\Lambda^\dagger\right)|0\rangle = \prod_{\boldsymbol{k}}\left(1 + \phi_{\boldsymbol{k}} c^\dagger_{\boldsymbol{k}\uparrow} c_{-\boldsymbol{k}\downarrow}\right)|0\rangle$$

where we see that the exponential simplifies because we are looking at fermionic raising and lowering operators. Notice how we have let $\boldsymbol{p} = 0$ for simplicity – it seems this still describes the physics quite well. Notice how this state has the peculiar property that

$$\langle\psi_{\mathrm{BCS}}|c_{-\boldsymbol{k}\downarrow}c_{\boldsymbol{k}\uparrow}|\psi_{\mathrm{BCS}}\rangle \sim \phi_{\boldsymbol{k}} \neq 0$$

The question now arises: what kind of a Hamiltonian describes a system where $|\psi_{\mathrm{BCS}}\rangle$ is an eigenstate. The answer is the BCS Hamiltonian:

$$H = \sum_{\boldsymbol{k}\sigma}\varepsilon_{\boldsymbol{k}\sigma}c^\dagger_{\boldsymbol{k}\sigma}c_{\boldsymbol{k}\sigma} + \sum_{\boldsymbol{kk'}} V_{\boldsymbol{k},\boldsymbol{k'}}c^\dagger_{\boldsymbol{k}\uparrow}c^\dagger_{-\boldsymbol{k}\downarrow}c_{-\boldsymbol{k'}\downarrow}c_{\boldsymbol{k'}\uparrow}$$

$V_{\boldsymbol{k},\boldsymbol{k}'}$ can take on a whole variety of different forms, however commonly one uses

$$V_{\boldsymbol{k},\boldsymbol{k}'} = \begin{cases} -\frac{g_0}{\mathcal{V}} & |\varepsilon_{\boldsymbol{k}}| < \omega_D \\ 0 & \text{otherwise} \end{cases}$$

This is the *s-wave manifestation*¶. We can write this in a form that it is ready for what is to come – a Hubbard-Stratonovich transformation:

$$H = \sum_{|\varepsilon_{\boldsymbol{k}}|<\omega_D,\sigma} \varepsilon_{\boldsymbol{k}} c^\dagger_{\boldsymbol{k}\sigma} c_{\boldsymbol{k}\sigma} - \frac{g_0}{V} A^\dagger A$$

where $A = \sum_{|\varepsilon_{\boldsymbol{k}}|<\omega_D} c_{-\boldsymbol{k}\downarrow} c_{\boldsymbol{k}\uparrow}$.

D. Mean-Field Description of the Condensate

As we saw, the BCS-ground state exhibits so-called *off-diagonal-long-range order*, which is the fact that Equation III C is non-zero. The pair operator A is extensive, and in the superconducting state $\langle A \rangle \propto \mathcal{V}$, however, the pair *density*

$$\Delta = |\Delta| e^{i\phi} = -\frac{g_0}{\mathcal{V}} \langle A \rangle$$

is intensive. This value, Δ, is the *order parametre* of superconductivity and contrary to other order parametres it has both a modulus and a phase. The fact that it is an order parametre is quite intuitive: above the critical temperature (*i.e* when are we not in the superconducting state) there is no off-diagonal-long-range order, whereas below the critical temperature there is. It turns out that $\Delta \sim \sqrt{T_C - T}$ close to T_C as one may expect from an order parametre.

If we express $A = \langle A \rangle + \delta A = \Delta + \delta A$ then we obtain that

$$-\frac{g_0}{\mathcal{V}} A^\dagger A = \bar{\Delta} A + A^\dagger \Delta + \mathcal{V} \frac{\bar{\Delta}\Delta}{g_0} + \frac{g_0}{\mathcal{V}} \delta A^\dagger \delta A$$

it turns out that the last term can be ignored in the thermodynamic limit, as it is $\mathcal{O}(1)$ whereas the other terms are $\mathcal{O}(\mathcal{V})$, which means that in the thermodynamic limit we can rewrite the BCS Hamiltonian as a mean-field Hamiltonian and not omit any important physics:

$$H_{\text{MFT}} = \sum_{\boldsymbol{k}\sigma} \varepsilon_{\boldsymbol{k}} c^\dagger_{\boldsymbol{k}\sigma} c_{\boldsymbol{k}\sigma} + \sum_{\boldsymbol{k}} \left[\bar{\Delta} c_{-\boldsymbol{k}\downarrow} c_{\boldsymbol{k}\uparrow} + c^\dagger_{\boldsymbol{k}\uparrow} c^\dagger_{-\boldsymbol{k}\downarrow} \Delta \right] + \frac{\mathcal{V}}{g_0} \bar{\Delta}\Delta$$

where Δ is to be determined self-consistently by minimising the free energy.

¶ Sometimes papers refer to s-wave superconductors and this is what is meant.

E. Physical Pictures of BCS Theory

1. *Pairs as Spins*

The pairing term in the mean-field Hamiltonian:

$$H_{\text{P}}(\boldsymbol{k}) = \bar{\Delta} c_{-\boldsymbol{k}\downarrow} c_{\boldsymbol{k}\uparrow} + c^\dagger_{\boldsymbol{k}\uparrow} c^\dagger_{-\boldsymbol{k}\downarrow} \Delta$$

can on the one hand be thought of as the following

$$e^- + e^- \leftrightharpoons \text{pair}^{2-}$$

however, it can also be beneficial to instead think of it as the creation of a cooper pair and a hole (we move one of the electrons to the right-hand-side of the process above), so we instead think of $c_{-\boldsymbol{k}\downarrow} = h^\dagger_{\boldsymbol{k}}$, which means we write the pairing term as

$$H_{\text{P}}(\boldsymbol{k}) = \bar{\Delta} h^\dagger_{\boldsymbol{k}} c_{\boldsymbol{k}\uparrow} + c^\dagger_{\boldsymbol{k}\uparrow} h_{\boldsymbol{k}} \Delta$$

where now

$$e^- \leftrightharpoons \text{pair}^{2-} + h^+$$

This process is referred to as *Andreev Reflection.* Following this we should associate the particle energy with $\varepsilon_{\boldsymbol{k}}$ but the hole energy with $-\varepsilon_{-\boldsymbol{k}}$. The dispersions intersect precisely at the Fermi-surface, which means that the mixing of these states (which occurs due to the Andreev Reflection) will open a gap, eliminating the Fermi-Surface. That is, in the particle-hole basis

$$\psi_{\boldsymbol{k}} = \begin{pmatrix} c_{\boldsymbol{k}\uparrow} \\ h_{\boldsymbol{k}\downarrow} \end{pmatrix}$$

the Hamiltonian reads

$$H = \begin{pmatrix} \varepsilon_{\boldsymbol{k}} & \Delta \\ \bar{\Delta} & -\varepsilon_{\boldsymbol{k}} \end{pmatrix} \quad \rightsquigarrow \quad E_{\boldsymbol{k}} = \sqrt{\varepsilon^2_{\boldsymbol{k}} + |\Delta|^2}$$

2. *Nambu Spinors*

The basis I referred to as the particle-hole basis is actually called a *Nambu Spinor*:

$$\psi_{\boldsymbol{k}} = \begin{pmatrix} c_{\boldsymbol{k}\uparrow} \\ c^\dagger_{-\boldsymbol{k}\downarrow} \end{pmatrix}, \qquad \psi^\dagger_{\boldsymbol{k}} = \begin{pmatrix} c^\dagger_{\boldsymbol{k}\uparrow} & c_{-\boldsymbol{k}\downarrow} \end{pmatrix}$$

which allows us to write the full Hamiltonian as

$$H = \sum_{\boldsymbol{k}} \psi^\dagger_{\boldsymbol{k}} (\boldsymbol{h}_{\boldsymbol{k}} \cdot \boldsymbol{\tau}) \psi_{\boldsymbol{k}} + \mathcal{V} \frac{\bar{\Delta}\Delta}{g_0}$$

where

$$\boldsymbol{h_k} = \begin{pmatrix} \mathrm{Re}\Delta \\ -\mathrm{Im}\Delta \\ \varepsilon_{\boldsymbol{k}} \end{pmatrix} = \begin{pmatrix} \Delta_1 \\ \Delta_2 \\ \varepsilon_{\boldsymbol{k}} \end{pmatrix}$$

and $\boldsymbol{\tau}$ is the vector of Pauli-Matrices. Thus we see that $\boldsymbol{h_k}$ behaves like a Zeeman field, but in isospin space.

3. *Anderson's Domain-Wall Interpretation of BCS-Theory*

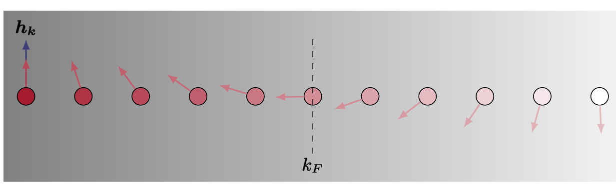
Let us define an isospin operator $\boldsymbol{\tau_k} \equiv \psi_{\boldsymbol{k}}^\dagger \boldsymbol{\tau} \psi_{\boldsymbol{k}}$, which has the properties of a spin-$\frac{1}{2}$ operator acting in charge space. The z-component can be written out as

$$\tau_{3\boldsymbol{k}} = n_{\boldsymbol{k}\uparrow} + n_{-\boldsymbol{k}\downarrow} - 1$$

Thus we can define its eigenvectors as up- and down-spin states in isospin space:

$$\begin{aligned} \tau_{3\boldsymbol{k}} \left|\Uparrow_{\boldsymbol{k}}\right\rangle &= \left|\Uparrow_{\boldsymbol{k}}\right\rangle, \qquad & \left|\Uparrow_{\boldsymbol{k}}\right\rangle &= \left|2\right\rangle = c_{\boldsymbol{k}\uparrow}^\dagger c_{-\boldsymbol{k}\downarrow}^\dagger \left|0\right\rangle \\ \tau_{3\boldsymbol{k}} \left|\Downarrow_{\boldsymbol{k}}\right\rangle &= -\left|\Downarrow_{\boldsymbol{k}}\right\rangle, \qquad & \left|\Downarrow_{\boldsymbol{k}}\right\rangle &= \left|0\right\rangle \end{aligned}$$

Notice how in a normal metal $\Delta = 0$, so only τ_3 appears in Equation III E 2, so we are in the up-state below the Fermi-surface, but in the down-state above the Fermi-surface. In the superconducting phase the transition from fully up to fully down will be softened, as is shown below



Figur 2: *The isospin along the z-axis is represented as the the vertical component of the arrow. Inside the Fermi surface the up-state is occupied, outside the Fermi surface it is the down-state that is occupied*

In a normal metal a quasiparticle arises when we flip the isospin of a state, for instance if we are outside the Fermi surface we can make a quasiparticle by exciting a particular state into the up-state. In superconductors we define a *Bogoliobov Quasiparticle* as the state where we have flipped the isospin of a particle, however, now there are also x and y components that are flipped.

As we saw, the quasiparticle energy is $E_{\boldsymbol{k}} = \sqrt{\varepsilon_{\boldsymbol{k}}^2 + |\Delta|^2}$, which allows us to define a unit vector in isospin space

$$\hat{n}_{\boldsymbol{k}} = \frac{1}{E_{\boldsymbol{k}}} \begin{pmatrix} \Delta_1 \\ \Delta_2 \\ \varepsilon_{\boldsymbol{k}} \end{pmatrix}$$

The $U(1)$ symmetry of our system allows us to freely choose a phase of Δ, so we can effectively eliminate Δ_2. Thus the arrows in Figure 2 are the nonzero components of $\hat{n}_{\boldsymbol{k}}$. We can define

$$\cos\theta_{\boldsymbol{k}} \equiv \frac{\varepsilon_{\boldsymbol{k}}}{E_{\boldsymbol{k}}} = \langle \tau_{3\boldsymbol{k}} \rangle, \qquad \sin\theta_{\boldsymbol{k}} \equiv \frac{\Delta}{E_{\boldsymbol{k}}} = \langle \tau_{1\boldsymbol{k}} \rangle$$

Notice how

$$\sin\theta_{\boldsymbol{k}} = \langle c_{\boldsymbol{k}\uparrow}^\dagger c_{-\boldsymbol{k}\downarrow}^\dagger + c_{-\boldsymbol{k}\downarrow} c_{\boldsymbol{k}\uparrow} \rangle = 2\langle c_{\boldsymbol{k}\uparrow}^\dagger c_{-\boldsymbol{k}\downarrow}^\dagger \rangle$$

because we have chosen $\langle \tau_{2\boldsymbol{k}} \rangle = \langle c_{\boldsymbol{k}\uparrow}^\dagger c_{-\boldsymbol{k}\downarrow}^\dagger - c_{-\boldsymbol{k}\downarrow} c_{\boldsymbol{k}\uparrow} \rangle = 0$ and hence Equation III D becomes

$$\Delta = g_0 \int_{|\varepsilon_{\boldsymbol{k}}| < \omega_D} \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3} \frac{\Delta}{2\sqrt{\varepsilon_{\boldsymbol{k}}^2 + \Delta^2}}$$

which is the *BCS Gap Equation* at zero temperature. If we assume that the density of states is constant around the Fermi-level (very common assumption for superconductivity) we get

$$1 = g_0 N(0) \int_{-\omega_D}^{\omega_D} \mathrm{d}\varepsilon \, \frac{1}{2\sqrt{\varepsilon^2 + \Delta^2}} = g_0 N(0) \sinh^{-1}\left(\frac{\omega_D}{\Delta}\right)$$

$$\approx g_0 N(0) \ln\left(\frac{2\omega_D}{\Delta}\right)$$

so, in the superconducting ground state, the BCS gap is given by

$$\Delta = 2\omega_D \exp\left(-\frac{1}{g_0 N(0)}\right)$$

notice the disappearance of the factor 2, which we saw in Equation III B.

F. The BCS Ground State

We would now like to construct a state whose isospin is always aligned with the Weiss-field ($\boldsymbol{h_k}$). This is done by simply taking the linear combination**

$$|\mathrm{BCS}\rangle = \prod_{\boldsymbol{k}} \left(\cos\frac{\theta_{\boldsymbol{k}}}{2} + \sin\frac{\theta_{\boldsymbol{k}}}{2} c_{-\boldsymbol{k}\downarrow}^\dagger c_{\boldsymbol{k}\uparrow}^\dagger\right) |0\rangle$$

** This is a normalised version of the coherent state discussed previously the coefficients are commonly written as

$$|\mathrm{BCS}\rangle = \prod_{\boldsymbol{k}} \left(u_{\boldsymbol{k}} + v_{\boldsymbol{k}} c^{\dagger}_{-\boldsymbol{k}\downarrow} c^{\dagger}_{\boldsymbol{k}\uparrow} \right) |0\rangle$$

where

$$u_{\boldsymbol{k}} = \sqrt{\frac{1}{2}\left[1 + \frac{\varepsilon_{\boldsymbol{k}}}{E_{\boldsymbol{k}}}\right]}, \qquad v_{\boldsymbol{k}} = \sqrt{\frac{1}{2}\left[1 - \frac{\varepsilon_{\boldsymbol{k}}}{E_{\boldsymbol{k}}}\right]}$$

Notice how the BCS-ground state breaks the $U(1)$-symmetry which the BCS-Hamiltonian possesses, let $c^{\dagger}_{\boldsymbol{k}\sigma} \to e^{i\alpha} c^{\dagger}_{\boldsymbol{k}\sigma}$, then (ignoring the normalisation):

$$|\mathrm{BCS}\rangle \to |\alpha\rangle = \prod_{\boldsymbol{k}} \left(1 + e^{2i\alpha} \phi_{\boldsymbol{k}} c^{\dagger}_{\boldsymbol{k}\uparrow} c_{-\boldsymbol{k}\downarrow}\right) |0\rangle = \sum \frac{e^{i2n\alpha}}{\sqrt{n!}} |n\rangle$$

which means that $\Delta = -\frac{g_0}{\mathcal{V}} \sum_{\boldsymbol{k}} \langle \alpha | c_{-\boldsymbol{k}\downarrow} c_{\boldsymbol{k}\uparrow} | \alpha \rangle$ obtains a phase:

$$\Delta \to e^{2i\alpha} \Delta$$

Additionally we have that

$$\hat{N} |\alpha\rangle = \sum \frac{1}{\sqrt{n!}} 2n e^{i2n\alpha} |\alpha\rangle = -i \frac{\mathrm{d}}{\mathrm{d}\alpha} |\alpha\rangle$$

So that $\hat{N} \equiv -i \frac{\mathrm{d}}{\mathrm{d}\alpha}$. This is exactly like the relationship between the momentum and position operators.

G. Quasiparticle Excitations in BCS Theory

Let us diagonalise the BCS-Hamiltonian

$$H = \sum_{\boldsymbol{k}} \psi^{\dagger}_{\boldsymbol{k}} (\boldsymbol{h}_{\boldsymbol{k}} \cdot \boldsymbol{\tau}) \psi_{\boldsymbol{k}} + \mathcal{V} \frac{\bar{\Delta}\Delta}{g_0}$$

We are interested in diagonalised the first term (Δ is just a number), which we can do by writing this term using the previously defined vector $\hat{n}_{\boldsymbol{k}}$:

$$\boldsymbol{h}_{\boldsymbol{k}} \cdot \boldsymbol{\tau} = E_{\boldsymbol{k}} \hat{n}_{\boldsymbol{k}} \cdot \boldsymbol{\tau}$$

so we are interested in the eigenvectors of $\hat{n}_{\boldsymbol{k}} \cdot \boldsymbol{\tau}$:

$$\hat{n}_{\boldsymbol{k}} \cdot \tau \begin{pmatrix} u_{\boldsymbol{k}} \\ v_{\boldsymbol{k}} \end{pmatrix} = \begin{pmatrix} u_{\boldsymbol{k}} \\ v_{\boldsymbol{k}} \end{pmatrix}, \qquad \hat{n}_{\boldsymbol{k}} \cdot \tau \begin{pmatrix} -v^*_{\boldsymbol{k}} \\ u^*_{\boldsymbol{k}} \end{pmatrix} = -\begin{pmatrix} -v^*_{\boldsymbol{k}} \\ u^*_{\boldsymbol{k}} \end{pmatrix}$$

thus by defining the unitary matrix

$$U_{\boldsymbol{k}} = \begin{pmatrix} u_{\boldsymbol{k}} & -v^*_{\boldsymbol{k}} \\ v_{\boldsymbol{k}} & u^*_{\boldsymbol{k}} \end{pmatrix}$$

we can diagonalise the Hamiltonian

$$U^{\dagger}_{\boldsymbol{k}} (\hat{n}_{\boldsymbol{k}} \cdot \boldsymbol{\tau}) U_{\boldsymbol{k}} = \tau_3$$

This transformation is a *Bogoliubov Transformation*, which we encountered in CMT1:

$$\begin{pmatrix} a_{\boldsymbol{k}\uparrow} \\ a^{\dagger}_{-\boldsymbol{k}\downarrow} \end{pmatrix} = \begin{pmatrix} u^*_{\boldsymbol{k}} & v^*_{\boldsymbol{k}} \\ -v_{\boldsymbol{k}} & u_{\boldsymbol{k}} \end{pmatrix} \begin{pmatrix} c_{\boldsymbol{k}\uparrow} \\ c^{\dagger}_{-\boldsymbol{k}\downarrow} \end{pmatrix}$$

In this basis the Hamiltonian reads

$$H = \sum_{\boldsymbol{k}} E_{\boldsymbol{k}} \left(a^{\dagger}_{\boldsymbol{k}\uparrow} a_{\boldsymbol{k}\uparrow} - a_{-\boldsymbol{k}\downarrow} a^{\dagger}_{-\boldsymbol{k}\downarrow} \right) + \mathcal{V} \frac{\bar{\Delta}\Delta}{g_0}$$

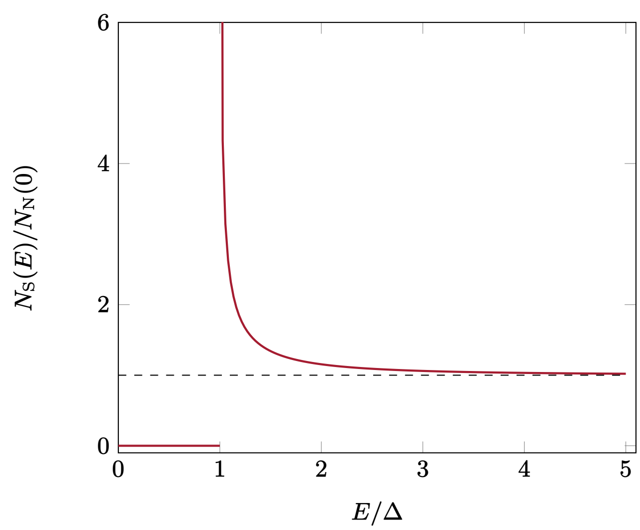
which we can write in a more familiar form

$$H = \sum_{\boldsymbol{k}\sigma} E_{\boldsymbol{k}} \left(a^{\dagger}_{\boldsymbol{k}\sigma} a_{\boldsymbol{k}\sigma} - \frac{1}{2} \right) + \mathcal{V} \frac{\bar{\Delta}\Delta}{g_0}$$

We can obtain the density of Bogoliubov quasiparticles per spin through the relation $N_{\mathrm{S}}(E)\mathrm{d}E = N_{\mathrm{N}}(0)\mathrm{d}\,|\varepsilon|$, where $N_{\mathrm{N}}(0) = 2N(0)$:

$$N_{\mathrm{S}}(E_{\boldsymbol{k}}) = N_{\mathrm{N}}(0) \frac{\mathrm{d}\,|\varepsilon_{\boldsymbol{k}}|}{\mathrm{d}E_{\boldsymbol{k}}} = N_n(0) \left(\frac{E_{\boldsymbol{k}}}{\sqrt{E^2_{\boldsymbol{k}} - |\Delta|^2}} \right) \theta(E - |\Delta|)$$

which is plotted below



Figur 3: *BCS Theory's prediction of the density of states of a superconductor with a square root discontinuity at $E = \Delta$: the coherence peak*

H. Path Integral Formulation

As anticipated the BCS-path integral looks like

$$Z = \int \mathcal{D}[\bar{c}, c] e^{-S}, \quad S = \int_0^\beta \mathrm{d}\tau \sum_{\boldsymbol{k}\sigma} \bar{c}_{\boldsymbol{k}\sigma} \left(\partial_\tau + \varepsilon_{\boldsymbol{k}}\right) c_{\boldsymbol{k}\sigma} - \frac{g_0}{\mathcal{V}} \bar{A} A$$

When dealing with superconductivity we must always keep in mind that $|\varepsilon_{\boldsymbol{k}}| < \omega_D$. The Hubbard-Stratonovich transformation removes the quartic term:

$$-g\bar{A}A \to \bar{\Delta}A + \bar{A}\Delta + \frac{\mathcal{V}}{g_0}\bar{\Delta}\Delta$$

where $\Delta(\tau)$ are fluctuating complex fields:

$$Z = \int \mathcal{D}[\Delta, c] e^{-S},$$
$$S = \int_0^\beta \mathrm{d}\tau \sum_{\boldsymbol{k}\sigma} \bar{c}_{\boldsymbol{k}\sigma} \left(\partial_\tau + \varepsilon_{\boldsymbol{k}}\right) c_{\boldsymbol{k}\sigma} + \bar{\Delta}A + \bar{A}\Delta + \frac{\mathcal{V}}{g_0}\bar{\Delta}\Delta$$

As expected this is expressed nicely with Nambu-Spinors

$$Z = \int \mathcal{D}[\Delta, \psi] e^{-S},$$
$$S = \int_0^\beta \mathrm{d}\tau \sum_{\boldsymbol{k}\sigma} \bar{\psi}_{\boldsymbol{k}} \left(\partial_\tau + \underline{h}_{\boldsymbol{k}}\right) \psi_{\boldsymbol{k}} + \frac{\mathcal{V}}{g_0}\bar{\Delta}\Delta$$

where $h_{\boldsymbol{k}} = \varepsilon_{\boldsymbol{k}}\tau_3 + \Delta_1\tau_1 + \Delta_2\tau_2$ and $\Delta = \Delta_1 - i\Delta_2$. We can integrate out the fermionic fields:

$$Z = \int \mathcal{D}[\bar{\Delta}, \Delta] e^{-S_{\mathrm{E}}[\bar{\Delta}, \Delta]},$$
$$e^{-S_{\mathrm{E}}[\bar{\Delta}, \Delta]} = \prod_{\boldsymbol{k}} \det[\partial_\tau + \underline{h}_{\boldsymbol{k}}(\tau)] e^{-\mathcal{V}\int_0^\beta \mathrm{d}\tau \frac{\bar{\Delta}\Delta}{g_0}}$$

and so

$$S_{\mathrm{E}}[\bar{\Delta}, \Delta] = \mathcal{V}\int_0^\beta \mathrm{d}\tau \frac{\bar{\Delta}\Delta}{g_0} - \sum_{\boldsymbol{k}} \mathrm{tr}\, \ln[\partial_\tau + \underline{h}_{\boldsymbol{k}}]$$

As we discussed previously, for superconductors in the thermodynamic limit we can effectively replace the Hamiltonian with its mean-field equivalent, because the δA terms disappeared for sufficiently large volumes. Thus we can treat Δ as its expectation value, which we will assume is independent of τ. Thus the mean field partition function $Z \approx Z_{\mathrm{BCS}} = e^{-S_{\mathrm{E}}[\bar{\Delta}_0, \Delta_0]}$ is

$$Z_{\mathrm{BCS}} = \prod_{\boldsymbol{k}} \det[\partial_\tau + \underline{h}_{\boldsymbol{k}}] \exp\left(-\frac{\mathcal{V}\beta}{g_0}\bar{\Delta}\Delta\right)$$

where we have omitted the 0-subscript on Δ for brevity. As usual it is easier if we work in Matsubara-Fourier space

$$Z_{\mathrm{BCS}} = \int \prod_{\boldsymbol{k}n} \mathrm{d}\bar{\psi}_{\boldsymbol{k}n} \mathrm{d}\psi_{\boldsymbol{k}n} e^{-S_{\mathrm{MFT}}[\bar{\psi}_{\boldsymbol{k}n}, \psi_{\boldsymbol{k}n}]}$$
$$S_{\mathrm{MFT}}[\bar{\psi}_{\boldsymbol{k}n}, \psi_{\boldsymbol{k}n}] = \sum_{\boldsymbol{k}n} \bar{\psi}_{\boldsymbol{k}n}(-i\omega_n + \underline{h}_{\boldsymbol{k}})\psi_{\boldsymbol{k}n} + \beta\mathcal{V}\frac{\bar{\Delta}\Delta}{g_0}$$

where $\psi_{\boldsymbol{k}}(\tau) = \frac{1}{\sqrt{\beta}}\sum_n \psi_{\boldsymbol{k}n} e^{-i\omega_n\tau}$. We can integrate out the fermionic fields to obtain

$$Z_{\mathrm{BCS}} = \prod_{\boldsymbol{k}n} \left[\omega_n^2 + \varepsilon_{\boldsymbol{k}}^2 + |\Delta|^2\right] e^{-\frac{\beta\mathcal{V}|\Delta|^2}{g_0}} = e^{-S_{\mathrm{E}}}$$

and hence we have an expression for the free energy

$$F[\Delta, T] = -T\sum_{\boldsymbol{k}n} \ln\left[\omega_n^2 + \varepsilon_{\boldsymbol{k}}^2 + |\Delta|^2\right] + \mathcal{V}\frac{|\Delta|^2}{g_0}$$

1. BCS Gap Equations: Revisited

The free energy is minimised when $\frac{\partial F}{\partial \bar{\Delta}} = 0$, which gives us

$$\frac{1}{g_0} = \frac{1}{\beta\mathcal{V}}\sum_{\boldsymbol{k}n} \frac{1}{\omega_n^2 + E_{\boldsymbol{k}}^2}$$

which is the first BCS Gap Equation, but we can do better than this, by performing the Matsubara sum.

$$\frac{1}{\beta}\sum_n \frac{1}{\omega_n^2 + E_{\boldsymbol{k}}^2} \xrightarrow[\text{Integral}]{\text{Contour}} \frac{\tanh\left(\frac{\beta E_{\boldsymbol{k}}}{2}\right)}{2E_{\boldsymbol{k}}}$$

and so we obtain the second BCS gap equation

$$\frac{1}{g_0} = \int_{|\varepsilon_{\boldsymbol{k}}|<\omega_D} \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3} \left[\frac{\tanh(\beta E_{\boldsymbol{k}}/2)}{2E_{\boldsymbol{k}}}\right]$$

I. Nambu-Gor'kov Green's Function

Let us define a Green's function (matrix) which we make by taking the time-ordered expectation value of two Nambu-spinors:

$$\underline{\mathcal{G}}_{\alpha\beta}(\boldsymbol{k}, \tau) = -\langle \mathbb{T}\, \psi_{\boldsymbol{k}\alpha}(\tau)\psi^\dagger_{\boldsymbol{k}\beta}(0)\rangle$$

One often calls the off-diagonal components F:

$$F(\boldsymbol{k}, \tau) = -\langle \mathbb{T}\, c_{\boldsymbol{k}\uparrow}(\tau)c_{-\boldsymbol{k}\downarrow}(0)\rangle,$$
$$\bar{F}(\boldsymbol{k}, \tau) = -\langle \mathbb{T}\, c^\dagger_{-\boldsymbol{k}\downarrow}(\tau)c^\dagger_{\boldsymbol{k}\uparrow}(0)\rangle$$

Notice how these are non-zero for the superconducting state, due to the off-diagonal-long-range order. We can write this out in Matsubara space

$$\underline{\mathcal{G}}(k) = \frac{1}{(i\omega_n)^2 - E_{\boldsymbol{k}}^2}\left(i\omega_n + \Delta_1\tau_1 + \Delta_2\tau_2 + \varepsilon_{\boldsymbol{k}}\tau_3\right)$$

This can be obtained by our previous methods or by using the Dyson Equation where the self-energy is given by the sum over even numbers of Andreev Reflections.

Note that the diagonal terms are

$$\operatorname{diag}\mathcal{G}_k = \frac{1}{(i\omega_n)^2 - E_{\boldsymbol{k}}^2}\begin{pmatrix} i\omega_n + \varepsilon_{\boldsymbol{k}} \\ i\omega_n - \varepsilon_{\boldsymbol{k}} \end{pmatrix}$$

Notice how

$$\frac{1}{2}\left(\frac{1}{i\omega_n + E_{\boldsymbol{k}}} + \frac{1}{i\omega_n - E_{\boldsymbol{k}}}\right) = \frac{i\omega_n}{(i\omega_n)^2 - E_{\boldsymbol{k}}^2}$$
$$-\frac{\varepsilon_{\boldsymbol{k}}}{2E_{\boldsymbol{k}}}\left(\frac{1}{i\omega_n + E_{\boldsymbol{k}}} - \frac{1}{i\omega_n - E_{\boldsymbol{k}}}\right) = \frac{\varepsilon_{\boldsymbol{k}}}{(i\omega_n)^2 - E_{\boldsymbol{k}}^2}$$

which means that

$$\frac{i\omega_n + \varepsilon_{\boldsymbol{k}}}{(i\omega_n)^2 - E_{\boldsymbol{k}}^2} = \frac{v_{\boldsymbol{k}}^2}{i\omega_n + E_{\boldsymbol{k}}} + \frac{u_{\boldsymbol{k}}^2}{i\omega_n - E_{\boldsymbol{k}}}$$
$$\frac{i\omega_n - \varepsilon_{\boldsymbol{k}}}{(i\omega_n)^2 - E_{\boldsymbol{k}}^2} = \frac{u_{\boldsymbol{k}}^2}{i\omega_n + E_{\boldsymbol{k}}} + \frac{v_{\boldsymbol{k}}^2}{i\omega_n - E_{\boldsymbol{k}}}$$

which tells us that

$$\operatorname{diag}\mathcal{G}_k = \begin{pmatrix} u_{\boldsymbol{k}}^2 & v_{\boldsymbol{k}}^2 \\ v_{\boldsymbol{k}}^2 & u_{\boldsymbol{k}}^2 \end{pmatrix}\begin{pmatrix} (i\omega - E_{\boldsymbol{k}})^{-1} \\ (i\omega + E_{\boldsymbol{k}})^{-1} \end{pmatrix}$$

1. *Tunneling Density of States and Coherence Factors*

Let us consider the particle part of the Nambu-Gor'kov Green's function, so $\mathcal{G}_{11}$. If we analytically continue it and take the imaginary part we get

$$A(\boldsymbol{k},\omega) = \frac{1}{\pi}\operatorname{Im}\mathcal{G}_{11}(\boldsymbol{k},\omega - i\delta) = u_{\boldsymbol{k}}^2\delta(\omega - E_{\boldsymbol{k}}) + v_{\boldsymbol{k}}^2\delta(\omega + E_{\boldsymbol{k}})$$

so we see that $u_{\boldsymbol{k}}^2$ describes the probability to create a positive energy quasiparticle, whereas $v_{\boldsymbol{k}}^2$ describes the probability to create a negative energy state.

J. From BCS To Ginzburg-Landau

To derive Ginzburg-Landau theory, let us rewrite Equation III H in a way that we can easily expand in orders of the order-parametre:

$$S[\bar{\Delta},\Delta] = \mathcal{V}\int_0^\beta \mathrm{d}\tau \frac{\bar{\Delta}\Delta}{g_0} - \operatorname{tr}\ln\left[-\underline{G}_0^{-1}(1 + \underline{G}_0\underline{\Delta})\right]$$

where

$$-\underline{G}_0^{-1} = -i\omega_n + \varepsilon_{\boldsymbol{k}}\tau_3, \quad \underline{\Delta} = \Delta_1\tau_1 + \Delta_2\tau_2$$

now, assuming that Δ is a small parametre, then we can expand the logarithm as

$$-\operatorname{tr}\ln[1 + \underline{G}_0\underline{\Delta}] \approx \underbrace{-\operatorname{tr}[\underline{G}_0\underline{\Delta}]}_{=0} + \frac{1}{2}\operatorname{tr}[\underline{G}_0\underline{\Delta}\underline{G}_0\underline{\Delta}] + \mathcal{O}(\Delta^4)$$

let's ignore the fourth order term and calculate the second order term. It turns out that

$$\frac{1}{2}\operatorname{tr}[\underline{G}_0\underline{\Delta}\underline{G}_0\underline{\Delta}] = \frac{1}{\beta\mathcal{V}}\sum_{kk'} G_{0\uparrow}(k)\Delta_q G_{0\downarrow}(k')\bar{\Delta}_{-q}$$

so up to fourth order in Δ the effective action is

$$S[\bar{\Delta},\Delta] = \sum_q \underbrace{\left(\frac{1}{g_0} - \frac{1}{\beta V}\sum_k G_0(k)G_0(q-k)\right)}_{2r(q)}|\Delta_q|^2 + \frac{1}{4}|\Delta|^4$$

where u comes from $\operatorname{tr}[G\Delta G\Delta G\Delta G\Delta]$ terms. Thus by assuming that Δ is small we have obtained Ginzburg-Landau theory, which tells us a lot about the physics of superconductivity, however, instead of it being a lucky guess, we have derived it from a microscopic theory.

K. Anderson-Higgs Mechanism in BCS-Theory

The full Nambu-Gor'kov Green's function is

$$\underline{\tilde{G}}^{-1} = -\tau_0\partial_\tau - \tau_1\Delta_0 - \tau_3\left(e\tilde{\phi} + \frac{1}{2m}\left(-i\nabla - e\tau_3\tilde{\boldsymbol{A}}\right)^2 - \mu\right)$$

where we have gauged Δ's phase out by means of a unitary transformation, which leaves the action unchanged. The transformation however let

$$\Delta \to \Delta_0 = |\Delta|$$
$$\boldsymbol{A} \to \boldsymbol{A} - e^{-1}\nabla\theta \equiv \tilde{\boldsymbol{A}}$$
$$\phi \to \phi + ie^{-1}\partial_\tau\theta \equiv \tilde{\phi}$$

Let us split this up into three parts depending on which order of $\tilde{A}_\mu$ appears:

$$\tilde{\underset{\sim}{G}}^{-1} = \tilde{\underset{\sim}{G}}_0^{-1} - \underset{\sim}{\chi}_1 - \underset{\sim}{\chi}_2$$

where

$$\begin{aligned}\tilde{\underset{\sim}{G}}_0^{-1} &\equiv -\tau_0\partial_\tau - \tau_3\left(\frac{(-i\nabla)^2}{2m} - \mu\right) - \tau_1\Delta_0 \\ \tilde{\underset{\sim}{\chi}}_1 &\equiv \tau_3 e\tilde{\phi} - \frac{e}{2m}\tau_0\left[-i\nabla, \tilde{\boldsymbol{A}}\right]_+ \\ \tilde{\underset{\sim}{\chi}}_2 &\equiv \tau_3\frac{e^2\tilde{A}^2}{2m}\end{aligned}$$

the action can be expanded using $\ln(1-x) \approx -x - \frac{1}{2}x^2$:

$$\begin{aligned}S &= -\operatorname{tr}\ln[-\tilde{\underset{\sim}{G}}^{-1}] \\ &\approx -\operatorname{tr}\ln[-\tilde{\underset{\sim}{G}}_0^{-1}] + \operatorname{tr}\ln[\tilde{\underset{\sim}{G}}_0\underset{\sim}{\chi}_1] + \operatorname{tr}\ln[\tilde{\underset{\sim}{G}}_0\underset{\sim}{\chi}_2] \\ &\quad + \frac{1}{2}\operatorname{tr}\ln[(\tilde{\underset{\sim}{G}}_0\underset{\sim}{\chi}_1)^2]\end{aligned}$$

1. *First Order in the Electromagnetic-Field*

The first order term

$$\begin{aligned}S^{(1)} &= \frac{1}{\beta\mathcal{V}}\sum_k \operatorname{tr}\left[\tilde{\underset{\sim}{G}}_0(k)\underset{\sim}{\chi}_1(k,k)\right] \\ &= \frac{e}{\beta\mathcal{V}}\sum_k \operatorname{tr}\left[\tilde{\underset{\sim}{G}}_0(k)\left(\tau_3\phi_0 - m^{-1}\tau_0\boldsymbol{k}\cdot\boldsymbol{A}_0\right)\right]\end{aligned}$$

where we have used that $A_\mu = \tilde{A}_\mu$ at $q = 0$. $\tilde{\underset{\sim}{G}}_0^{-1}$ is even in $\boldsymbol{k}$, which implies that the $\boldsymbol{k}\cdot\boldsymbol{A}_0$ term integrates to zero leaving only

$$\begin{aligned}S^{(1)} &= \frac{e\phi_0}{\beta\mathcal{V}}\sum_k \operatorname{tr}\left[\tilde{\underset{\sim}{G}}_0(k)\tau_3\right] \\ &= \frac{e\phi_0}{\mathcal{V}}\sum_{\boldsymbol{k}}\frac{1}{\beta}\sum_{i\omega_n}\left(\frac{i\omega_n + \varepsilon_k}{(i\omega_n)^2 - E_k^2} - \frac{i\omega_n - \varepsilon_k}{(i\omega_n)^2 - E_k^2}\right)\end{aligned}$$

performing the Matsubara sum gives us $S^{(1)} \approx en\int \mathrm{d}\tau\mathrm{d}\boldsymbol{r}\phi(\boldsymbol{r},\tau)$, which is cancelled by the corresponding term from the positive ionic background – we are assuming that the superconductor is neutral.

2. *Second Order in the Electromagnetic-Field*

There are two second order terms, the first of which is quite simple as the calculation is exactly what we did, but replacing $\phi_0 \rightsquigarrow \frac{eA_0}{2m}$:

$$\operatorname{tr}\left[\tilde{\underset{\sim}{G}}_0\underset{\sim}{\chi}_2\right] = \frac{ne^2}{2m}A_0^2$$

That is the *diamagnetic term*. Now for the paramagnetic term

$$\begin{aligned}\frac{1}{2}\operatorname{tr}\left[\tilde{\underset{\sim}{G}}_0\underset{\sim}{\chi}_1\tilde{\underset{\sim}{G}}_0\underset{\sim}{\chi}_1\right] &= \frac{1}{2\beta}\sum_{kq}\operatorname{tr}\left[\tilde{\underset{\sim}{G}}_0(k)\tau_3(e\tilde{\phi}_q)\tilde{\underset{\sim}{G}}_0(k)\tau_3(e\tilde{\phi}_{-q})\right] \\ &\quad + \frac{1}{m^2}\operatorname{tr}\left[\tilde{\underset{\sim}{G}}_0(k)\tilde{\underset{\sim}{G}}_0(k)(\boldsymbol{k}\cdot\tilde{\boldsymbol{A}}_{\boldsymbol{q}})(\boldsymbol{k}\cdot\tilde{\boldsymbol{A}}_{-\boldsymbol{q}})\right]\end{aligned}$$

where we've ignore the q in $\underset{\sim}{G}_0(k+q)$, because this just gives us an extra term that doesn't change the physics of the problem. This evaluates to

$$\begin{aligned}&\int \mathrm{d}\tau\mathrm{d}\boldsymbol{r}\ \left[-c_1e^2\tilde{\phi}^2(\boldsymbol{r},\tau) + c_2e^2\tilde{\boldsymbol{A}}^2(\boldsymbol{r},\tau)\right] \\ =&\int \mathrm{d}\tau\mathrm{d}\boldsymbol{r}\ \left[-c_1(\partial_\tau\theta - ie\phi)^2 + c_2(\nabla\theta - e\boldsymbol{A})^2\right]\end{aligned}$$

where $c_1 \equiv \nu_F$ and $c_2 \equiv \frac{n_s}{2m}$ and n_s is the superfluid density.

Now we assume that we are at temperatures high enough that we ignore the $\partial_\tau\theta$ term, and let us reintroduce the $(\nabla\times\boldsymbol{A})$ term which we in reality should have included from the beginning, then the action reads

$$\begin{aligned}S[\boldsymbol{A},\theta] &= \frac{\beta}{2}\int \mathrm{d}\boldsymbol{r}\left[\frac{n_s}{m}(\nabla\theta - e\boldsymbol{A})^2 + \frac{1}{\mu_0}(\nabla\times\boldsymbol{A})^2\right] \\ &= \frac{\beta}{2}\sum_{\boldsymbol{q}}\left[\frac{n_s}{m}(i\boldsymbol{q}\theta_{\boldsymbol{q}} - e\boldsymbol{A}_{\boldsymbol{q}})\cdot(-i\boldsymbol{q}\theta_{-\boldsymbol{q}} - e\boldsymbol{A}_{-\boldsymbol{q}})\right] + \\ &\quad \frac{1}{\mu_0}(\boldsymbol{q}\times\boldsymbol{A}_{\boldsymbol{q}})(\boldsymbol{q}\times\boldsymbol{A}_{-\boldsymbol{q}})\end{aligned}$$

Integrating out θ we get

$$S[\boldsymbol{A}] = \frac{\beta}{2}\sum_{\boldsymbol{q}}\left(\frac{n_se^2}{m} + \frac{q^2}{\mu_0}\right)\left(\boldsymbol{A}_{\boldsymbol{q}}\cdot\boldsymbol{A}_{-\boldsymbol{q}} - (\hat{\boldsymbol{q}}\cdot\boldsymbol{A}_{\boldsymbol{q}})(\hat{\boldsymbol{q}}\cdot\boldsymbol{A}_{-\boldsymbol{q}})\right)$$

by defining $\boldsymbol{A}_{\boldsymbol{q}}^\perp \equiv \boldsymbol{A}_{\boldsymbol{q}} - \hat{\boldsymbol{q}}(\hat{\boldsymbol{q}}\cdot\boldsymbol{A}_{\boldsymbol{q}})$ and using the Coulomb Gauge to remove the $\boldsymbol{A}_{\boldsymbol{q}}^\parallel \equiv \hat{\boldsymbol{q}}(\hat{\boldsymbol{q}}\cdot\boldsymbol{A}_{\boldsymbol{q}})$ part, we can write

$$S[\boldsymbol{A}] = \frac{\beta}{2}\sum_{\boldsymbol{q}}\boldsymbol{A}_{\boldsymbol{q}}^\perp\left(\frac{n_se^2}{m} + \frac{q^2}{\mu_0}\right)\boldsymbol{A}_{-\boldsymbol{q}}^\perp$$

see wee that the electromagnetic fields have gained mass, because the propagator is of the form $\mathcal{D}^{-1} = m^2 + |\boldsymbol{k}|^2$. This implies that there is attenuation of the electromagnetic field inside of superconductors, which is what leads to the Meissner Effect.

At the saddle point $\frac{\delta S}{\delta \boldsymbol{A}^{\perp}_{-\boldsymbol{q}}} = 0$ we have

$$\frac{\mu_0 n_s e^2}{m}\boldsymbol{A}^{\perp}_{\boldsymbol{q}} = -q^2 \boldsymbol{A}^{\perp}_{\boldsymbol{q}} \quad \rightsquigarrow \quad \mathcal{L}\{\}\, B = \lambda_L^{-2}\boldsymbol{B}$$

where $\lambda_L \equiv \sqrt{\frac{m}{\mu_0 n_s e^2}}$. This is the *First London Equation*, which we derived in CMP2 using Landau-Ginzburg Theory.

L. Are We Really Describing a Superconductor?

Our Free energy is of the form

$$F = \nu_F(\partial_\tau\theta - ie\phi)^2 + \frac{n_s}{2m}(\nabla\theta - e\boldsymbol{A})^2 + \frac{1}{\mu_0}(\nabla\times\boldsymbol{A})^2$$

From which we can calculate the current[††]

$$\boldsymbol{j} = \nabla_{\boldsymbol{A}} F = \frac{n_s}{m}(\nabla\theta - e\boldsymbol{A})$$

The derivative gives us

$$\frac{\partial \boldsymbol{j}}{\partial t} = \frac{e n_s}{m}\boldsymbol{E}$$

telling us that for a constant electric field the current rises linearly as a function of time which is only possible if there is no resistance.

IV. BOLTZMANN EQUATION

The Boltzmann Equation is an equation used in *kinetic theory*, which is the study of non-equilibrium phenomena in dilute gasses. In these dilute gasses particles follow Hamiltonian dynamics but suffer occasional scattering from other particles or impurities.

The central object in kinetic theory is the phase-space distribution function $f(\boldsymbol{r},\boldsymbol{p},t)$. The quantity

$$f(\boldsymbol{r},\boldsymbol{p},t)\frac{\mathrm{d}\boldsymbol{r}\mathrm{d}\boldsymbol{p}}{(2\pi\hbar)^3} = f(\boldsymbol{r},\boldsymbol{p},t)\mathrm{d}\boldsymbol{r}\mathrm{d}\boldsymbol{\pi}$$

describes the mean number of particles in a phase-space volume of $\mathrm{d}\boldsymbol{r}\mathrm{d}\boldsymbol{\pi}$ at time t. We have introduced[‡‡] $\mathrm{d}\boldsymbol{\pi}$ to avoid the factor of $(2\pi\hbar)^3$. In equilibrium the function $f(\boldsymbol{r},\boldsymbol{p},t)$ can be the Maxwell-Boltzmann distribution, the Fermi-Dirac distribution or the Bose-Einstein distribution or even something more exotic, depending on what the system is that you are describing.

The *Boltzmann Equation* is the continuity equation in phase space preserving the number of particles:

$$\frac{\partial f}{\partial t} + \nabla_{\boldsymbol{r}}\cdot(\dot{\boldsymbol{r}}f) + \nabla_{\boldsymbol{k}}\cdot(\dot{\boldsymbol{k}}f) = 0$$

this is the collisionless Boltzmann Equation, to include collisions we introduce the so-called *collision integral* on the right-hand side

$$\frac{\partial f}{\partial t} + \nabla_{\boldsymbol{r}}\cdot(\dot{\boldsymbol{r}}f) + \nabla_{\boldsymbol{k}}\cdot(\dot{\boldsymbol{k}}f) = \left(\frac{\partial f}{\partial t}\right)_{\text{coll.}}$$

The collision integral is not a derivative, this is just notation; sometimes it is also written as $\mathcal{I}$. We should think of this as Hamiltonian evolution (lhs) with occasional collisions (rhs) that abruptly push you from one part of phase-space to another.

During the Hamiltonian evolution we have that $\nabla_{\boldsymbol{r}}\cdot\dot{\boldsymbol{r}} + \nabla_{\boldsymbol{k}}\cdot\dot{\boldsymbol{k}} = 0$ so

BOLTZMANN EQUATION

$$\frac{\partial f}{\partial t} + \dot{\boldsymbol{r}}\cdot\nabla_{\boldsymbol{r}} f + \dot{\boldsymbol{k}}\cdot\nabla_{\boldsymbol{k}} f = \left(\frac{\partial f}{\partial t}\right)_{\text{coll.}}$$

We can think of the Boltzmann Equation as the full-derivative of the distribution function f, which means that if we change variables we just need to adjust the names of the variables in the Boltzmann Equation, for instance if we want to change between canonical and kinetic momenta.

A. Collisions: Classical

Let us consider a classical scattering process in which we assume $\mathrm{d}\boldsymbol{r}$ is fixed. The probability of scattering during the time interval $\mathrm{d}t$ is $\mathrm{d}\boldsymbol{\pi} w(\boldsymbol{p}',\boldsymbol{p})\mathrm{d}t$, so the rate of change in f due to collisions is

$$\left(\frac{\partial f}{\partial t}\right)_{\text{coll.}} = -\int \mathrm{d}\boldsymbol{\pi}\ \left[w(\boldsymbol{p}',\boldsymbol{p})f(\boldsymbol{p}) - w(\boldsymbol{p},\boldsymbol{p}')f(\boldsymbol{p}')\right]$$

The 2 particle version thereof would be

$$\left(\frac{\partial f_1}{\partial t}\right)_{\text{coll.}} = -\int \mathrm{d}\boldsymbol{\pi}_2\mathrm{d}\boldsymbol{\pi}_1'\mathrm{d}\boldsymbol{\pi}_2'\ \big[w(\boldsymbol{p}_1',\boldsymbol{p}_2';\boldsymbol{p}_1,\boldsymbol{p}_2)f(\boldsymbol{p}_1)f(\boldsymbol{p}_2) - w(\boldsymbol{p}_1,\boldsymbol{p}_2;\boldsymbol{p}_1',\boldsymbol{p}_2')f(\boldsymbol{p}_1')f(\boldsymbol{p}_2')\big]$$

[††] $\nabla_{\boldsymbol{A}}(\nabla\times\boldsymbol{A}) = 0$ because $(\nabla\times\boldsymbol{A})_x$ only depends on A_y and A_z.

[‡‡] Jens uses $\mathrm{d}\tau$, but we already integrate over something called τ in other sections, and I don't like using the same letters for different things.

where we've assumed that the two-particle distribution function splits into the product of two one-particle distribution functions, which is valid in the low-density limit ($f(\boldsymbol{p}_1;\boldsymbol{p}_2) = f(\boldsymbol{p}_1)f(\boldsymbol{p}_2)$).

For time-reversal symmetric systems we have that

$$w(\boldsymbol{p}_1,\boldsymbol{p}_2;\boldsymbol{p}_1',\boldsymbol{p}_2') = w(-\boldsymbol{p}_1',-\boldsymbol{p}_2';-\boldsymbol{p}_1,-\boldsymbol{p}_2)$$

and for inversion-symmetric systems we have

$$w(\boldsymbol{p}_1,\boldsymbol{p}_2;\boldsymbol{p}_1',\boldsymbol{p}_2') = w(-\boldsymbol{p}_1,-\boldsymbol{p}_2;-\boldsymbol{p}_1',-\boldsymbol{p}_2')$$

if we have a system that is both time-reversal and inversion symmetric then it must hold that

$$w(\boldsymbol{p}_1,\boldsymbol{p}_2;\boldsymbol{p}_1',\boldsymbol{p}_2') = w(\boldsymbol{p}_1',\boldsymbol{p}_2';\boldsymbol{p}_1,\boldsymbol{p}_2)$$

which is referred to as the *Principle of Detailed Balance*. In such systems the collision integral simplifies to

$$\left(\frac{\partial f_1}{\partial t}\right)_{\text{coll.}} = \int \mathrm{d}\boldsymbol{\pi}_2 \mathrm{d}\boldsymbol{\pi}_1' \mathrm{d}\boldsymbol{\pi}_2' \; w(\boldsymbol{p}_1,\boldsymbol{p}_2;\boldsymbol{p}_1',\boldsymbol{p}_2') \left[f_1'f_2' - f_1f_2\right]$$

where I have omitted the $\boldsymbol{p}$s.

B. Boltzmann H-Theorem

Let us introduce

$$\rho_\phi(\boldsymbol{r},t) = \int \mathrm{d}\boldsymbol{\pi} \; \phi(\boldsymbol{p},f) f(\boldsymbol{r},\boldsymbol{p},t)$$

for some function(al) ϕ. Its rate of change due to 2-body collisions is[§§]

$$\left(\frac{\partial \rho_\phi}{\partial t}\right)_{\text{coll.}} = \int \mathrm{d}\boldsymbol{\pi}_1 \boldsymbol{\pi}_2 \boldsymbol{\pi}_1' \boldsymbol{\pi}_2' \; \bar{\phi} w(\boldsymbol{p}_1,\boldsymbol{p}_2;\boldsymbol{p}_1',\boldsymbol{p}_2') \left[f_1'f_2' - f_1f_2\right]$$

where we have defined $\bar{\phi} \equiv \phi + f\frac{\partial \phi}{\partial f}$. Manipulating the w term and using the principle of detailed balance we obtain

$$\left(\frac{\partial \rho_\phi}{\partial t}\right)_{\text{coll.}} = -\frac{1}{4}\int \mathrm{d}\boldsymbol{\pi}_1 \boldsymbol{\pi}_2 \boldsymbol{\pi}_1' \boldsymbol{\pi}_2' \; w(\boldsymbol{p}_1,\boldsymbol{p}_2;\boldsymbol{p}_1',\boldsymbol{p}_2') \times (\bar{\phi}_1 + \bar{\phi}_2 - \bar{\phi}_1' - \bar{\phi}_2')(f_1f_2 - f_1'f_2')$$

[§§] this seems like magic to me...

suppose $\phi = \ln f$

$$\left(\frac{\partial}{\partial t}\int \mathrm{d}\boldsymbol{\pi}_1 \; f_1 \ln f_1\right)_{\text{coll.}} = -\frac{1}{4}\int \mathrm{d}\boldsymbol{\pi}_1 \boldsymbol{\pi}_2 \boldsymbol{\pi}_1' \boldsymbol{\pi}_2' \; w(\boldsymbol{p}_1,\boldsymbol{p}_2;\boldsymbol{p}_1',\boldsymbol{p}_2') \times (f_1f_2 - f_1'f_2') \ln\left(\frac{f_1f_2}{f_1'f_2'}\right)$$

the right-hand side is always negative, which means that $H = \int \mathrm{d}\boldsymbol{\pi}_1 f_1 \ln f_1$ (enthalpy) always decreases due to collisions. Relating enthalpy to entropy tells us that entropy always increases due to collisions.

C. Conservation Laws

If ϕ is chosen to be independent of f then the density

$$\rho_\phi(\boldsymbol{r},t) = \int \mathrm{d}\boldsymbol{\pi} \; \phi(\boldsymbol{p}) f(\boldsymbol{r},\boldsymbol{p},t)$$

is unchanged by collisions if ϕ is a collision invariant (because then $\phi_1 + \phi_2 = \phi_1' + \phi_2'$). We have a number of invariants: energy, E, momentum, $\pm p$, and numbers. When we minimise the enthalpy we require $f_1f_2 = f_1'f_2'$, so $\ln f_1 + \ln f_2 = \ln f_1' + \ln f_2'$. So $\ln f$ is also a collision invariant, which implies we can express it as a linear combination of the aforementioned collision invariants:

$$\ln f = A + \boldsymbol{B}\cdot\boldsymbol{p} + CE$$

suppose $A = \beta\mu$, $\boldsymbol{B} = \beta\boldsymbol{u}$ (drift velocity) and $C = -\beta$, we get

$$f = \exp\left[-\beta(E - \boldsymbol{p}\cdot\boldsymbol{u} - \mu)\right]$$

D. Collisions: Quantum

In the quantum mechanical Boltzmann Equation we need to account for Pauli's Exclusion principle (for fermions). For bosons we acquire a similar factor which accounts for Bose-statistics which makes it seem like bosons "prefer"to be in same states. Thus we need to modify the $f_1f_2 - f_1'f_2'$ term in the Boltzmann Equation:

$$f_1f_2 - f_1'f_2' \;\rightsquigarrow\; f_1f_2(1+\zeta f_1')(1+\zeta f_2') - f_1'f_2'(1+\zeta f_1)(1+\zeta f_2)$$

where $\zeta = 1$ for bosons and $\zeta = -1$ for fermions.

In equilibrium this vanishes, which gives us

$$\ln\left(\frac{f_1}{1+\zeta f_1}\right) + \ln\left(\frac{f_2}{1+\zeta f_2}\right) = \ln\left(\frac{f_1'}{1+\zeta f_1'}\right) + \ln\left(\frac{f_2'}{1+\zeta f_2'}\right)$$

So if $\ln\left(\frac{f}{1+\zeta f}\right) = x = \beta(E - \boldsymbol{p}\cdot\boldsymbol{u} - \mu)$ we get

$$\frac{f}{1+\zeta f} = e^x \quad \rightsquigarrow \quad f = \frac{1}{e^{-x} - \zeta}$$

giving us both the Bose-Einstein and the Fermi-Dirac distribution functions.

E. Linearisation

Consider the collision integral from Equation IV A:

$$\left(\frac{\partial f_1}{\partial t}\right)_{\text{coll.}} = \int \mathrm{d}\boldsymbol{\pi}_2 \mathrm{d}\boldsymbol{\pi}_1' \mathrm{d}\boldsymbol{\pi}_2'\; w(\boldsymbol{p}_1, \boldsymbol{p}_2; \boldsymbol{p}_1', \boldsymbol{p}_2')\,[f_1' f_2' - f_1 f_2]$$

Now let us assume that we are close to equilibrium, such that we can write $f = f^0 + f^0\psi$ for $\psi \ll 1$, where f^0 is the equilibrium distribution function. We have that $f_1^0 f_2^0 = f_{1'}^0 f_{2'}^0$ (the collision integral is zero in equilibrium), and means that up to first order in ψ we can write

$$\left(\frac{\partial f_1}{\partial t}\right)_{\text{coll.}} = \int \mathrm{d}\boldsymbol{\pi}_2 \mathrm{d}\boldsymbol{\pi}_1' \mathrm{d}\boldsymbol{\pi}_2'\; w(\boldsymbol{p}_1, \boldsymbol{p}_2; \boldsymbol{p}_1', \boldsymbol{p}_2')\times f_1^0\,[\psi_1' + \psi_2' - \psi_1 - \psi_2]$$

Additionally splitting the left-hand side up and using that f_0 is not a function of time we get that

$$\left(\frac{\partial \psi_1}{\partial t}\right)_{\text{coll.}} = \int \mathrm{d}\boldsymbol{\pi}_2 \mathrm{d}\boldsymbol{\pi}_1' \mathrm{d}\boldsymbol{\pi}_2'\; w(\boldsymbol{p}_1, \boldsymbol{p}_2; \boldsymbol{p}_1', \boldsymbol{p}_2')\times [\psi_1' + \psi_2' - \psi_1 - \psi_2]$$

So we have linearised the collision integral. Because the Boltzmann Equation is now a linear integro-differential equation, there is a lot of theory that we learned about in Quantum Mechanics that can be used. For instance there exists a basis of functions that span the general solution to the integro-differential equation. However, keep in mind that this only holds for small values of ψ.

F. Relaxation-Time Approximation

In the relaxation time approximation we assume that the collision integral is given by¶¶

$$\left(\frac{\partial f}{\partial t}\right)_{\text{coll.}} = -\frac{f - f^0}{\tau}$$

where τ is the relaxation time of the system. If you start a system at some $f \neq f^0$ then you can expect to wait $\sim \tau$ before the system returns to equilibrium.

Consider now the Boltzmann Equation where we ignore the $\left(\frac{\partial f}{\partial t}\right)$ and $\nabla_{\boldsymbol{k}}$ terms and use the relaxation-time approximation:

$$\boldsymbol{v}\cdot\nabla_{\boldsymbol{r}} f = -\frac{f - f_0}{\tau}$$

assume that f^0 only depends on $\beta(E-\mu)$, which means that the left-hand side becomes

$$\boldsymbol{v}\cdot\nabla_{\boldsymbol{r}} f \approx \boldsymbol{v}\cdot\nabla_{\boldsymbol{r}} f^0 = -\frac{\partial f^0}{\partial E}\boldsymbol{v}\cdot(E\beta\nabla T - \mu\beta\nabla T + \nabla\mu)$$

the last two terms can be written in terms of the enthalpy per particle, h, reducing it to

$$\boldsymbol{v}\cdot\nabla_{\boldsymbol{r}} f \approx f^0(\boldsymbol{v}\cdot\nabla T)k_B\beta^2(E - h)$$

whereas the collision integral becomes

$$-\frac{f - f^0}{\tau} = -\frac{f^0\psi}{\tau}$$

giving us

$$\psi = -k_B\beta^2\tau(E - h)\boldsymbol{v}\cdot\nabla T$$

using which we can calculate the heat current due to the gradient in the temperature

$$\boldsymbol{j}_e = \int \boldsymbol{\pi}\; \boldsymbol{v}(\boldsymbol{v}\cdot\nabla T)k_B\beta^2\tau(h - E)f^0 E = -\kappa\nabla T$$

where we can evaluate the thermal conductivity

$$\kappa = \int \mathrm{d}\boldsymbol{\pi}\; \frac{v^2}{3}E\tau(E - h)f^0\beta = \frac{5nk_B^2 T}{2m}$$

(the final equality holds if we use the Maxwell Distribution for f^0)

V. NON-EQUILIBRIUM KELDYSH FIELD THEORY

A. Closed Time Contour

Despite the fact that non-equilibrium Keldysh Field Theory considers real-time values, it is general enough to encompass descriptions of non-zero temperature systems.

¶¶ Jens Paaske's note here is "*One must be careful not to break conservation laws!*"

We consider a system whose state at $t = -\infty$ is *known* and is given by the density matrix $\hat{\rho}(-\infty)$. We assume that the Hamiltonian is time-dependent and that $\hat{H}(-\infty)$ describes a system of non-interacting particles. At finite times the interactions are turned on adiabatically, such that they have reached their actual value before we perform our "measurement". The density matrix evolves according to the Von Neumann Equation

$$\partial_t \hat{\rho}(t) = -i[\hat{H}(t), \hat{\rho}(t)]$$

because we know the state of system at $t = -\infty$ and we can calculate the time-evolution operator

$$\hat{\mathcal{U}}_{t,t'} = \mathbb{T}\,\exp\left(-i\int_{t'}^{t} \mathrm{d}t''\; H(t'')\right)$$

we can solve the Von Neumann Equation by simply evolving the state

$$\hat{\rho}(t) = \hat{\mathcal{U}}_{t,-\infty}\hat{\rho}(-\infty)\hat{\mathcal{U}}^{\dagger}_{t,-\infty}$$

We are often interested in expectation values of an operator, $\hat{\mathcal{O}}$:

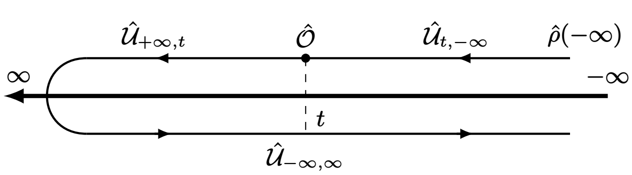
$$\langle\mathcal{O}\rangle(t) \equiv \frac{\mathrm{tr}\left[\hat{\mathcal{O}}\hat{\rho}(t)\right]}{\mathrm{tr}[\hat{\rho}(t)]} = \frac{\mathrm{tr}\left[\hat{\mathcal{U}}_{-\infty,t}\hat{\mathcal{O}}\hat{\mathcal{U}}_{t,-\infty}\hat{\rho}(-\infty)\right]}{\mathrm{tr}[\hat{\rho}(t)]}$$

where we have used the cyclicity of the trace and that $\hat{\mathcal{U}}^{\dagger}_{t,t'} = \hat{\mathcal{U}}_{t',t}$. This is also the starting point of equilibrium field theory, however, there one assumes that $\hat{\mathcal{U}}_{\infty,\infty}\,|0\rangle = e^{iL}\,|0\rangle$, which is what leads to the Gell-Mann-Low Formula see [? , Eq. 5.97].

For non-equilibrium field theory, however, we proceed by inserting $\mathbb{1} = \hat{\mathcal{U}}_{t,+\infty}\hat{\mathcal{U}}_{+\infty,t}$ just to the left of $\hat{\mathcal{O}}$ such that we obtain

$$\langle\mathcal{O}\rangle(t) = \frac{\mathrm{tr}\left[\hat{\mathcal{U}}_{-\infty,+\infty}\hat{\mathcal{U}}_{+\infty,t}\hat{\mathcal{O}}\hat{\mathcal{U}}_{t,-\infty}\hat{\rho}(-\infty)\right]}{\mathrm{tr}[\hat{\rho}(t)]}$$

This describes the evolution along the *closed time contour*, $\mathcal{C}$, which we refer to as the *Keldysh Contour*. Notice how we could have inserted the $\mathbb{1}$ just to the *right* of $\hat{\mathcal{O}}$, which would have put the observable on the backward branch of the contour, see Figure ??.



Figur 4: *The Keldysh Contour, $\mathcal{C}$ [? , Fig. 1.1]*

In Keldysh Field Theory it is customary to define the partition function as

$$Z = \langle\mathbb{1}\rangle = \frac{\mathrm{tr}\left[\hat{\mathcal{U}}_{\mathcal{C}}\hat{\rho}(-\infty)\right]}{\mathrm{tr}[\hat{\rho}(t)]} = 1$$

where we have defined $\hat{\mathcal{U}}_{\mathcal{C}} = \hat{\mathcal{U}}_{-\infty,+\infty}\hat{\mathcal{U}}_{+\infty,\infty}$ which is unity if the Hamiltonian is the same on the forward and backward portion of the contour. Thus we will often work with $Z = 1$, which means that we cannot derive physical quantities directly from it, as one would in statistical physics; despite this we will gain a rich understand of non-equilibrium field theory.

Let us define $\hat{H}^{\pm}_{V}(t) = \hat{H}(t) \pm \hat{\mathcal{O}}V(t)$ where the plus (minus) means that we use this expression for $\hat{H}_V$ on the forward (backward) branch. Since we have constructed a Hamiltonian that is not the same on the two branches our $Z[V]$ becomes a non-trivial functional of V:

$$Z[V] = \frac{\mathrm{tr}\left[\hat{\mathcal{U}}_{\mathcal{C}}[V]\hat{\rho}(-\infty)\right]}{\mathrm{tr}[\hat{\rho}(t)]}$$

this is the *generating functional.* From here we can obtain

$$\langle\hat{\mathcal{O}}\rangle(t) = \frac{i}{2}\frac{\delta Z[V]}{\delta V(t)}\bigg|_{V=0}$$

which is reminiscent of the corresponding expression in equilibrium many-body physics.

B. Bosons

Let us begin with a single bosonic Harmonic Oscillator

$$\hat{H} = \omega_0 \hat{b}^{\dagger}\hat{b}$$

and let us assume that the initial density matrix was

$$\hat{\rho}_{-\infty} = \exp\left(-\beta(\hat{H} - \mu\hat{N})\right) = \exp\left(-\beta(\omega_0 - \mu)\hat{N}\right)$$

Our Hamiltonian is the same on both parts of the contour, so $Z = 1$:

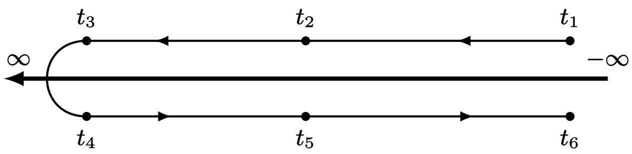
$$Z = \frac{\mathrm{tr}\left[\hat{\mathcal{U}}_{\mathcal{C}}\hat{\rho}_{-\infty}\right]}{\mathrm{tr}[\hat{\rho}_{-\infty}]}$$

let us calculate the denominator

$$\mathrm{tr}[\hat{\rho}_{-\infty}] = \sum_{n=0}^{\infty} e^{-\beta(\omega_0-\mu)n} = [1 - \rho(\omega_0)]^{-1},$$

$$\hat{\rho}(\omega_0) = \exp[-\beta(\omega_0 - \mu)]$$

let us discretise the Keldysh Contour that we can calculate the numerator by hand:



Figur 5: *A Discretised Keldysh Contour, $\mathcal{C}$ [? , Fig. 1.1]*

Define $\Delta t = t_{j+1} - t_j$, and now expand the numerator

$$\mathrm{tr}\Big[\hat{\mathcal{U}}_{\mathcal{C}}\hat{\rho}_{-\infty}\Big] \sim \langle\phi_6|\hat{\mathcal{U}}_{-\Delta t}|\phi_5\rangle\,\langle\phi_5|\hat{\mathcal{U}}_{-\Delta t}|\phi_4\rangle\,\langle\phi_4|\mathbb{1}|\phi_3\rangle\times \langle\phi_3|\hat{\mathcal{U}}_{\Delta t}|\phi_2\rangle\,\langle\phi_2|\hat{\mathcal{U}}_{\Delta t}|\phi_1\rangle\,\langle\phi_1|\hat{\rho}_{-\infty}|\phi_6\rangle$$

We are considering coherent states, so as long as the Hamiltonian is normal ordered we can use

$$\langle\phi_j|\hat{\mathcal{U}}_{\pm\Delta t}|\phi_{j-1}\rangle \approx e^{\bar{\phi}_j\phi_{j-1}}e^{\mp iH[\bar{\phi}_j,\phi_{j-1}]\Delta}$$

where we've assumed Δt is sufficiently small that we can use

$$\begin{aligned}\langle\phi_i|\exp\Big(a\hat{H}\Big)|\phi_j\rangle &\approx \langle\phi_i|1 + H[\bar{\phi}_i,\phi_j]|\phi_j\rangle \\ &\approx \langle\phi_i|\exp\Big(a\hat{H}[\bar{\phi}_i,\phi_j]\Big)|\phi_j\rangle\end{aligned}$$

The part containing $\hat{\rho}_{-\infty}$:

$$\langle\phi_1|\hat{\rho}_{-\infty}|\phi_6\rangle = e^{\bar{\phi}_1\phi_6\rho(\omega_0)}$$

Resulting in

$$Z = [1-\rho(\omega_0)]\int\prod_{j=1}^{2N}\mathrm{d}[\bar{\phi}_j,\phi_j]\exp\left(i\bar{\boldsymbol{\phi}}\cdot \underset{\sim}{G}^{-1}\cdot\boldsymbol{\phi}\right)$$

where in our case $N = 3$ and

$$i\underset{\sim}{G}^{-1} = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & \rho(\omega_0) \\ h_- & -1 & 0 & 0 & 0 & 0 \\ 0 & h_- & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & h_+ & -1 & 0 \\ 0 & 0 & 0 & 0 & h_+ & -1 \end{pmatrix}$$

and we have defined $h_\pm \equiv 1 \pm i\omega_0\Delta t$. We can evaluate the Gaussian integral giving us

$$\frac{1}{\det(-i\underset{\sim}{G}^{-1})} = \frac{1}{1-\rho(\omega_0)(h_-h_+)^{N-1}} \xrightarrow{N\to\infty} \frac{1}{1-\rho(\omega_0)}$$

so we see that $Z = 1$ once we take the $N \to \infty$ limit. Let us write things more explicitly

$$Z = \int \mathcal{D}[\bar{\phi}(t),\phi(t)]\exp\big(iS[\bar{\phi},\phi]\big)$$

where $\mathcal{D}[\bar{\phi}(t),\phi(t)] = \frac{1}{\mathrm{tr}\,\hat{\rho}_{-\infty}}\prod_{j=1}^{2N}\mathrm{d}[\bar{\phi}_j,\phi_j]$ and the action is given by

$$\begin{aligned}S[\bar{\phi},\phi] =& \sum_{j=2}^{2N}\delta t_j\left[i\bar{\phi}_j\left(\frac{\phi_j-\phi_{j-1}}{\delta t_j}\right) - \omega_0\bar{\phi}_j\phi_{j-1}\right] \\ &+ i\bar{\phi}_1[\phi_1 - \rho(\omega_0)\phi_{2N}] + i\bar{\phi}_{N+1}[\phi_{N+1}-\phi_N]\end{aligned}$$

where we've defined $\delta t_j = \pm(t_j - t_{j-1})$ (the $\pm$ depends on which branch of the contour we are on). Notice the similarity between the first term and Equation I B, however, the remaining terms were not seen in our previous formalism. Ignoring the latter terms we can write the first term in the action as

$$S_G[\bar{\phi},\phi] = \oint_{\mathcal{C}}\mathrm{d}t\;\bar{\phi}(t)G^{-1}\phi(t), \qquad G^{-1} = i\partial_t - \omega_0$$

We now split $\phi(t)$ up into its forward-branch part (ϕ^+) and it's backward-branch part (ϕ^-), which means we can write

$$S_G[\bar{\phi},\phi] = \int_{-\infty}^{\infty}\mathrm{d}t\;\begin{pmatrix}\bar{\phi}^+(t) & \bar{\phi}^-(t)\end{pmatrix}\tau_3(i\partial_t - \omega_0)\begin{pmatrix}\phi^+(t)\\ \phi^-(t)\end{pmatrix}$$

as it stands $\phi^\pm$ are completely independent, but in reality they are linked by the latter terms in Equation ??.

1. *Green's Functions*

Let us now calculate Green's functions, which we can do by inverting Equation ??:

$$i\underset{\sim}{G} = \frac{1}{g}\begin{pmatrix} 1 & h_-h_+^2\rho & h_+^2\rho & h_+^2\rho & h_+\rho & \rho \\ h_- & 1 & h_-h_+^2\rho & h_-h_+^2\rho & h_-h_+\rho & h_-\rho \\ h_-^2 & h_- & 1 & h_-^2h_+^2\rho & h_-^2h_+\rho & h_-^2\rho \\ h_-^2 & h_- & 1 & 1 & h_-^2h_+\rho & h_-^2\rho \\ h_-^2h_+ & h_-h_+ & h_+ & h_+ & 1 & h_-^2h_+\rho \\ h_-^2h_+^2 & h_-h_+^2 & h_+^2 & h_+^2 & h_+ & 1 \end{pmatrix}$$

where $g \equiv \det\big[-i\underset{\sim}{G}^{-1}\big]$ and $\rho \equiv \rho(\omega_0)$. These can be split up into the lesser than, greater than, time-ordered and anti time-ordered Green's functions, that evaluate to

$$\begin{aligned}\langle\phi^+(t)\bar{\phi}^-(t')\rangle &= iG^<_{t,t'} = n_\mathrm{B}(\omega_0)\exp\left(-i\omega_0(t-t')\right) \\ \langle\phi^-(t)\bar{\phi}^+(t')\rangle &= iG^>_{t,t'} = (1+n_\mathrm{B}(\omega_0))\exp\left(-i\omega_0(t-t')\right) \\ \langle\phi^+(t)\bar{\phi}^+(t')\rangle &= iG^{\mathbb{T}}_{t,t'} = i\theta(t-t')G^>_{t,t'} + i\theta(t'-t)G^<_{t,t'} \\ \langle\phi^-(t)\bar{\phi}^-(t')\rangle &= iG^{\tilde{\mathbb{T}}}_{t,t'} = i\theta(t-t')G^<_{t,t'} + i\theta(t'-t)G^>_{t,t'}\end{aligned}$$

Remember we are still looking at the single bosonic level in "thermal equilibrium"(in the sense that its occupations are given by the Bose-Einstein distribution)

Clearly these four functions are interrelated, which can also be seen in the following

$$G^{\mathbb{T}}_{t,t'} + G^{\tilde{\mathbb{T}}}_{t,t'} - G^{>}_{t,t'} - G^{<}_{t,t'} = -i\delta_{t,t'}$$

hence there are only three independent Green's functions. This is seen by changing basis; we perform a *Keldysh Rotation*, which is a unitary transformation:

$$\begin{pmatrix} \phi^{\mathrm{cl}} \\ \phi^{\mathrm{q}} \end{pmatrix} = \frac{1}{\sqrt{2}}(\tau_1 + \tau_3) \begin{pmatrix} \phi^{+} \\ \phi^{-} \end{pmatrix}$$

The transformed Green's matrix is

$$i\underline{G}_{t,t'} = \begin{pmatrix} iG^{\mathrm{K}}_{t,t'} & iG^{\mathrm{R}}_{t,t'} \\ iG^{\mathrm{A}}_{t,t'} & 0 \end{pmatrix}$$

where at $t \neq t'$ it holds that

$$\begin{aligned} G^{\mathrm{K}} &= G^{<} + G^{>} \\ G^{\mathrm{R}} &= \theta(t-t')(G^{>} - G^{<}) \\ G^{\mathrm{A}} &= \theta(t'-t)(G^{<} - G^{>}) \end{aligned}$$

Fourier transforming

$$\begin{aligned} G^{\mathrm{K}}(\omega) &= -2\pi i[2n_{\mathrm{B}}(\omega) + 1]\delta(\omega - \omega_0) \\ G^{\mathrm{R}}(\omega) &= \frac{1}{\omega - \omega_0 + i0^{+}} \\ G^{\mathrm{A}}(\omega) &= \frac{1}{\omega - \omega_0 - i0^{+}} \end{aligned}$$

So in equilibrium we get

$$G^{\mathrm{K}}(\varepsilon) = \coth\left(\frac{\varepsilon - \mu}{2T}\right)[G^{\mathrm{R}}(\varepsilon) - G^{\mathrm{A}}(\varepsilon)]$$

This result also holds for other systems in *thermal equilibrium.*

It is important here to recognise the retarded and advanced Green's functions that we've known about since CMT1 (although we didn't call them that in the course), and to see that this new Green's function, the Keldysh Green's function, carries information about occupations. A common Ansatz for G^{K} is

$$\begin{aligned} G^{\mathrm{K}} &= G^{\mathrm{R}} \circ F - F \circ G^{\mathrm{A}}\,, \\ (F \circ G)_{t,t'} &\equiv \int_{-\infty}^{\infty} \mathrm{d}t''\, F(t,t'')G(t'',t') \end{aligned}$$

where F is the distribution function (equilibrium or non-equilibrium) – in the equilibrium case we had $F(\omega) = \coth\left(\frac{\omega-\mu}{2T}\right)$.

C. Keldysh Action

The inverse of the matrix in Equation ?? has a zero $(0,0)$-component, our notation in the course was

$$G^{-1} = \begin{pmatrix} 0 & G_{\mathrm{A}}^{-1} \\ G_{\mathrm{R}}^{-1} & G_{\mathrm{K}}^{-1} \end{pmatrix}$$

so the action is

$$S[\phi^{\mathrm{cl}}, \phi^{\mathrm{q}}] = \iint_{-\infty}^{\infty} \mathrm{d}t\mathrm{d}t' \begin{pmatrix} \bar{\phi}^{\mathrm{cl}} & \bar{\phi}^{\mathrm{q}} \end{pmatrix}_t \begin{pmatrix} 0 & G_{\mathrm{A}}^{-1} \\ G_{\mathrm{R}}^{-1} & G_{\mathrm{K}}^{-1} \end{pmatrix}_{tt'} \begin{pmatrix} \phi^{\mathrm{cl}} \\ \phi^{\mathrm{q}} \end{pmatrix}_{t'}$$

we see that if $\phi^{\mathrm{q}} = 0$ then $S = 0$, which is equivalent to saying that if H is the same on both branches of the contour $Z = 1$.

We have that

$$[G^{-1}_{\mathrm{R(A)}}]_{t,t'} = \begin{cases} \delta(t-t')(i\vec{\partial}_{t'} - \omega_0 \pm i0^{+}) & \text{totheright} \\ \delta(t-t')(-i\overleftarrow{\partial}_{t} - \omega_0 \pm i0^{+}) & \text{totheleft} \end{cases}$$

And we have that

$$G_{\mathrm{K}}^{-1} = G_{\mathrm{R}}^{-1} \circ F - F \circ G_{\mathrm{A}}^{-1}$$

which means that in equilibrium we can write

$$G^{-1} = \begin{pmatrix} 0 & \omega - \omega_0 - i0^{+} \\ \omega - \omega + i0^{+} & 2i0^{+}\coth\left(\frac{\omega-\mu}{2T}\right) \end{pmatrix}$$

D. Feynman Diagrams

The Keldysh, retarded and advanced Green's functions can be represented in terms of Feynman Diagrams, in this course we used

$$G^{\mathrm{K}} =$$

$$G^{\mathrm{R}} =$$

$$G^{\mathrm{A}} =$$

E. From Keldysh to Langevin

Let us begin with the action for the single boson

$$S[\bar{\phi}, \phi] = \oint_{C} \mathrm{d}t\, \bar{\phi}(t) G^{-1} \phi(t), \quad G^{-1} = i\partial_t - \omega_0$$

and let us rewrite the complex fields in terms of their real and imaginary parts

$$\phi(t) = \frac{1}{\sqrt{2\omega_0}}\left(p(t) - i\omega_0 x(t)\right)$$

for $x(t), p(t) \in \mathbb{R}$. Putting this in for ϕ (and the conjugate for $\bar{\phi}$) and multiplying out we get

$$S[x,p] = \oint_C \mathrm{d}t \left[p\dot{x} - \frac{1}{2}p^2 - \frac{1}{2}\omega_0^2 x^2\right] + \text{const.}$$

where const. is the integral of a total derivative and hence doesn't affect the the equations of motion. Our partition function becomes

$$Z = \int \mathcal{D}[x,p] \exp\left(i \oint_C (p\dot{x} - H)\right)$$

notice how it is the Lagrangian we are integrating over – the Legendre transform of the Hamiltonian. Hence, generally we could write

$$Z = \int \mathcal{D}[x,p] \exp\left(i \oint_C (p\dot{x} - \frac{1}{2}p^2 - V(x))\right)$$

or, if we integrate out the momentum

$$S[x] = \oint_C \mathrm{d}t \left(\frac{1}{2}\dot{x}^2 - V(x)\right)$$

However, let us now split the contour into the forward and backwards branch and change variables to the classical and quantum fields:

$$S[x^{\mathrm{cl}}, x^{\mathrm{q}}] = \int_{-\infty}^{\infty} \mathrm{d}t \ \left(-2x^{\mathrm{q}}\ddot{x}^{\mathrm{cl}} - V(x^{\mathrm{cl}} + x^{\mathrm{q}}) + V(x^{\mathrm{cl}} - x^{\mathrm{q}})\right)$$

Suppose now we can treat x^{q} as small variables, such that we only need to expand the potential up to linear order in it:

$$S[x^{\mathrm{cl}}, x^{\mathrm{q}}] = \int_{-\infty}^{\infty} \mathrm{d}t \ 2x^{\mathrm{q}}\left(\ddot{x}^{\mathrm{cl}} + V'(x^{\mathrm{cl}})\right)$$

it turns out that $\int \mathcal{D}[x] e^{i\int \mathrm{d}t\, xf} = \delta(f)$ which means integrating out x^{q} gives us Newton's Second Law:

$$\ddot{x}^{\mathrm{cl}} = -V'(x^{\mathrm{cl}})$$

However, if x^{q} isn't a small parametre our formalism can account for the quantum fluctuations.

1. *Coupling the Harmonic Oscillator to a Bath*

Let us now include an interaction between the single boson level with a bath

$$S_{\mathrm{HO}}[x] = \int_{-\infty}^{\infty} \mathrm{d}t \ \left(-2x^{\mathrm{q}}\ddot{x}^{\mathrm{cl}} - V(x^{\mathrm{cl}} + x^{\mathrm{q}}) + V(x^{\mathrm{cl}} - x^{\mathrm{q}})\right)$$

$$S_{\mathrm{bath}}[\phi] = \frac{1}{2}\sum_s \int_{-\infty}^{\infty} \mathrm{d}t \ \bar{\phi}_s^T D_s^{-1}\phi_s$$

$$S_{\mathrm{int}}[x,\phi] = \sum_s g_s \int_{-\infty}^{\infty} \mathrm{d}t \ \bar{x}^T \tau_1 \phi_s$$

Let us integrate out the bath to obtain the dissipative action

$$S_{\mathrm{diss}} = \frac{1}{2}\iint_{-\infty}^{\infty} \mathrm{d}t\mathrm{d}t' \ \vec{x}^T(t)\mathbb{D}^{-1}(t-t')x(t')$$

where

$$\mathbb{D}^{-1}(t-t') = -\tau_1 \left[\sum_s g_s^2 D_s(t-t')\right]\tau_1$$

$\mathbb{D}$ has the same "causality structure"as we would expect from a Green's function, that is to say that it makes sense to assign a retarded, advanced and Keldysh component to it and that they all behave as they should. The retards and advanced components are

$$\mathbb{D}^{-1}_{\mathrm{R/A}} = \frac{1}{2}\sum_s \frac{g_s^2}{(\varepsilon \pm i0)^2 - \omega_s^2} = \int_0^{\infty} \frac{\mathrm{d}\omega}{2\pi}\frac{\omega J(\omega)}{\omega^2 - (\varepsilon \pm i0)^2}$$

where we've defined $J(\omega) \equiv \pi \sum_s \frac{g_s^2}{\omega_s}\delta(\omega - \omega_s)$. If we assume that $J(\omega) = 8\gamma\omega$ where γ is constant at small frequencies we find that

$$\begin{aligned}\mathbb{D}^{-1}_{\mathrm{R/A}}(\varepsilon) &= 4\gamma \int_0^{\infty} \frac{\mathrm{d}\omega}{2\pi}\frac{\omega^2}{\omega^2 - (\varepsilon \pm i0)^2} \\ &= \text{const.} \pm 2i\gamma\varepsilon\end{aligned}$$

which means that the Keldysh component is

$$\mathbb{D}^{-1}_{\mathrm{K}} = 4i\gamma\varepsilon \coth\left(\frac{\varepsilon}{2T}\right)$$

Thus we have a non-zero imaginary part now, which means there are non-zero lifetimes too. In t-space we get

$$\mathbb{D}^{-1}_{\mathrm{K}}(t-t') = 4i\gamma\left[(2T + C)\delta(t-t') - \frac{\pi T^2}{\sinh^2(\pi T(t-t'))}\right]$$

and $\mathbb{D}^{-1}_{\mathrm{R/A}} = \mp\gamma\delta(t-t')\partial_{t'}$. The resulting action is

$$S[x] = \int_{-\infty}^{\infty} \mathrm{d}t \; \left[-2x^{\mathrm{q}}(\ddot{x}^{\mathrm{cl}} + \gamma\dot{x}^{\mathrm{cl}}) - V(x^{\mathrm{cl}} + x^{\mathrm{q}}) + V(x^{\mathrm{cl}} - x^{\mathrm{q}})\right]$$
$$+2i\gamma \int_{-\infty}^{\infty} \mathrm{d}t \left[2T(x^{\mathrm{q}}(t))^2 + \frac{\pi T^2}{2}\int_{-\infty}^{\infty} \mathrm{d}t' \frac{[x^{\mathrm{q}}(t) - x^{\mathrm{q}}(t')]^2}{\sinh^2[\pi T(t-t')]}\right]$$

Now it is time to reinstate $\hbar$, we do so by dividing the entire action by $\hbar$ (so that $\exp(S/\hbar)$ has a unitless argument), but we also let $x^{\mathrm{q}} \to \hbar x^{\mathrm{q}}$ and $T \to T/\hbar$. Thus all linear terms in x^{q} aren't affected, we expand $V(x^{\mathrm{cl}} \pm x^{\mathrm{q}})$ up to linear order in x^{q} and we need to treat the last term separately when we let $\hbar \to 0$. We get

$$\lim_{\hbar\to 0} \frac{\pi T^2}{2\hbar \sinh^2(\pi T(t-t')/\hbar)} = T\delta(t-t')$$

which implies that the last term falls out completely leaving only

$$S[x] = \int_{-\infty}^{\infty} \mathrm{d}t \; \left[-2x^{\mathrm{q}}[\ddot{x}^{\mathrm{cl}} + \gamma\dot{x}^{\mathrm{cl}} + V'(x^{\mathrm{cl}})] + 4i\gamma T[x^{\mathrm{q}}(t)]^2\right]$$

we are almost ready to integrate x^{q} out again, but first we need to deal with the quadratic term. We do so by means of a Hubbard-Stratonovich transformation:

$$e^{-4\gamma T\int \mathrm{d}t [x^{\mathrm{q}}(t)]^2} = \int \mathcal{D}[\xi(t)] e^{-\int \mathrm{d}t [\frac{1}{4\gamma T}\xi^2(t) - 2i\xi(t)x^{\mathrm{q}}(t)]}$$

where the measure $\mathcal{D}[\xi(t)]$ is chosen such that $\int \mathcal{D}[\xi(t)] e^{-\int \mathrm{d}t[\frac{1}{4\gamma T}\xi^2(t)]} = 1$. The expectation value of an observable $\mathcal{O}[x^{\mathrm{cl}}]$ can be written as

$$\langle \mathcal{O}\rangle = \int \mathcal{D}[x^{\mathrm{q}}, x^{\mathrm{cl}}]\mathcal{O}[x^{\mathrm{cl}}]e^{iS[\boldsymbol{x}]}$$
$$= \int \mathcal{D}[\xi, x^{\mathrm{cl}}] e^{-\int \mathrm{d}t \frac{\xi^2}{4\gamma T}} \mathcal{O}[x^{\mathrm{cl}}]\delta(\ddot{x}^{\mathrm{cl}} + \gamma\dot{x}^{\mathrm{cl}} + V'(x^{\mathrm{cl}}) - \xi)$$

hence the only functions (trajectories) x^{cl} that contribute to $\langle\hat{\mathcal{O}}\rangle$ are the ones that satisfy the *Langevin Equation*

$$\ddot{x}^{\mathrm{cl}} = -\gamma\dot{x}^{\mathrm{cl}} - V'(x^{\mathrm{cl}}) + \xi(t)$$

which is Newton's Second Law with friction and with a *random* driving force, for which

$$\langle\xi(t)\xi(t')\rangle = \int \mathcal{D}[\xi]\xi(t)\xi(t')e^{-\int \mathrm{d}t \frac{\xi(t)}{4\gamma T}} = 2\gamma T\delta(t-t')$$

random forces of this type are often referred to as *white noise forces.*

F. From Keldysh to the Boltzmann Equation

Let us now begin by extending the Keldysh formalism to multiple bosons, *i.e* now

$$H_0 = \sum_{\boldsymbol{k}} \omega_{\boldsymbol{k}} \hat{b}^\dagger_{\boldsymbol{k}}\hat{b}_{\boldsymbol{k}}$$

where our initial density matrix is $\hat{\rho}_{-\infty} = \exp\left[-\beta(\hat{H}_0 - \mu\hat{N})\right]$ and $\hat{N} \equiv \sum_{\boldsymbol{k}} \hat{b}^\dagger_{\boldsymbol{k}}\hat{b}_{\boldsymbol{k}}$. Similarly to before we get

$$Z = \int \mathcal{D}[\phi^{\mathrm{cl}}, \phi^{\mathrm{q}}]e^{iS_0[\phi^{\mathrm{cl}},\phi^{\mathrm{q}}]} = 1$$

where

$$\mathcal{D}[\phi^{\mathrm{cl}}, \phi^{\mathrm{q}}] = \frac{1}{\mathrm{tr}[\rho_{-\infty}]}\prod_{\boldsymbol{k}}\prod_{j=1}^{N} \frac{\mathrm{dRe}\phi^{\mathrm{cl}}_j(\boldsymbol{k})\mathrm{dIm}\phi^{\mathrm{cl}}_j(\boldsymbol{k})}{\pi} \times \frac{\mathrm{dRe}\phi^{\mathrm{q}}_j(\boldsymbol{k})\mathrm{dIm}\phi^{\mathrm{q}}_j(\boldsymbol{k})}{\pi}$$

And now in the action we sum over momenta and $\omega_0 \to \omega_{\boldsymbol{k}}$, which means that the Green's functions are momentum dependent (so are all the ϕs). The Green's Functions are

$$G^{\mathrm{R}}_0(\boldsymbol{k}, t) = (\varepsilon - \omega_{\boldsymbol{k}} + i0^+)^{-1}$$
$$G^{\mathrm{A}}_0(\boldsymbol{k}, t) = (\varepsilon - \omega_{\boldsymbol{k}} - i0^+)^{-1}$$
$$G^{\mathrm{K}}_0(\boldsymbol{k}, t) = -2\pi i F(\varepsilon)\delta(\varepsilon - \omega_{\boldsymbol{k}})$$

Now introducing interactions

$$H_{\mathrm{int}} = \frac{1}{2}\sum_{qkk'} U_q b^\dagger_k b^\dagger_{k'} b_{k'+q} b_{k-q}$$

This is a normal-ordered Hamiltonian, which means that when it's inside the action we can just replace operators with ϕ and $\bar{\phi}$.

$$S_{\mathrm{int}} = -\frac{g}{2}\sum_{qkk'}\oint_{\mathcal{C}} \mathrm{d}t \; \bar{\phi}_{k,t}\bar{\phi}_{k',t}\phi_{k'+q,t}\phi_{k-q,t}$$

Splitting the contour into the forward and backward branch then performing the Keldysh rotation gives us

$$S_{\mathrm{int}} = -\frac{g}{2}\int_{-\infty}^{\infty} \mathrm{d}t \int \mathrm{d}\boldsymbol{r} \; [\bar{\phi}^{\mathrm{cl}}\bar{\phi}^{\mathrm{q}}\phi^{\mathrm{cl}}\phi^{\mathrm{cl}} + \bar{\phi}^{\mathrm{cl}}\bar{\phi}^{\mathrm{q}}\phi^{\mathrm{q}}\phi^{\mathrm{q}} + c.c.]$$

Hence the interaction introduces the following vertices

Figur 6: *Vertex introduced by the* $\bar{\phi}^{\mathrm{cl}}\bar{\phi}^{\mathrm{q}}\phi^{\mathrm{cl}}\phi^{\mathrm{cl}}$ *term*

Figur 7: *Vertex introduced by the* $\bar{\phi}^{\text{cl}}\bar{\phi}^{\text{q}}\phi^{\text{q}}\phi^{\text{q}}$ *term*

The two complex conjugates are just the time reversed processes, so where we swap the directions of all the arrows. These extra terms in the partition function do not contribute – all additional terms are zero.

1. *Keldysh-Dyson Equation*

The *full* Green's function is given by

$$\underline{G}^{\alpha\beta}(\boldsymbol{k},\boldsymbol{k}',t,t') = -i\int \mathcal{D}[\phi^{\text{cl}},\phi^{\text{q}}]\phi^{\alpha}(\boldsymbol{k},t)\bar{\phi}^{\beta}(\boldsymbol{k}',t')e^{iS[\phi^{\text{cl}},\phi^{\text{q}}]}$$

where $S = S_0 + S_{\text{int}}$. If we now expand in terms of S_{int} we can obtain a Dyson Equation:

$$\underline{G} = \underline{G}_0 + \underline{G}_0 \circ \underline{\Sigma} \circ \underline{G}$$

where the self-energy has the same causal structure as $\underline{G}_0^{-1}$, that is

$$\underline{\Sigma} = \begin{pmatrix} 0 & \Sigma^{\text{A}} \\ \Sigma^{\text{R}} & \Sigma^{\text{K}} \end{pmatrix}$$

Moving this around we obtain

$$(\underline{G}_0^{-1} - \underline{\Sigma}) \circ \underline{G} = \nVdash$$

We can write this out in terms of the components, starting with the retarded and advanced parts

$$(G_0^{-1} - \Sigma^{\text{R(A)}}) \circ G^{\text{R(A)}} = 1$$

written out

$$\left(i\partial_t + \frac{1}{2m}\mathcal{L}\{\}\boldsymbol{r} - V^{\text{cl}}(x) - \Sigma^{\text{R(A)}}\right) \circ G^{\text{R(A)}}_{x,x'} = \delta^{(4)}(x - x')$$

where $x = (\boldsymbol{r}, t)$. However, there is also a Keldysh component

$$(G_0^{-1} - \Sigma^{\text{R}}) \circ G^{\text{K}} - \Sigma^{\text{K}} \circ G^{\text{A}} = 0$$

using our Ansatz that $G^{\text{K}} = G^{\text{R}} \circ F - F \circ G^{\text{A}}$, after some manipulation we get that

$$[F\,[1pt]\circ, G_0^{-1}] = \Sigma^{\text{K}} - (\Sigma^{\text{R}} \circ F - F \circ \Sigma^{\text{A}})$$

written out we get the quantum kinetic equation for the distribution matrix, F:

$$-\left[\left(\partial_t + \frac{1}{2m}\mathcal{L}\{\}\boldsymbol{r} - V^{\text{cl}}(x)\right)[1pt]\circ, F\right] = \Sigma^{\text{K}} - (\Sigma^{\text{R}} \circ F - F \circ \Sigma^{\text{A}}$$

The left-hand side is the kinetic term – what was referred to as the *streaming term* for the Boltzmann Equation, whereas the right-hand side is the collision integral. In equilibrium the collision term disappears, which means that the Keldysh self-energy obtains the same form as our Ansatz for the Keldysh Green's function.

Let us write out the different components in the commutator

$$[i\partial_t\,[1pt]\circ, F] = (i\partial_t + i\partial_{t'})F(t,t')$$
$$[\mathcal{L}\{\}\boldsymbol{r}\,[1pt]\circ, F] = (\mathcal{L}\{\}\boldsymbol{r} - \mathcal{L}\{\}\boldsymbol{r}')F(\boldsymbol{r},\boldsymbol{r}')$$
$$[V(x)\,[1pt]\circ, F] = [V(x) - V(x')]F(x,x')$$

G. Wigner-Transformation

Let us consider a function of the form

$$A(x_1,x_2) = A(\boldsymbol{r}_1,t_1,\boldsymbol{r}_2,t_2)$$

Then the Wigner-Transformation is

$$A(x,p) \equiv \int \mathrm{d}x' e^{-ipx'} A(x + x'/2, x - x'/2)$$

where $x = \frac{1}{2}(x_1 + x_2)$ and $x' = x_1 - x_2$. The inverse transformation would be

$$A(x_1,x_2) = \sum_{\boldsymbol{p}} e^{ip(x_1 - x_2)} A\left(\frac{1}{2}(x_1 + x_2), p\right)$$

We will use this for the quantum kinetic equation, so it is useful to know what the Wigner-Transformation of a convolution is

$$C(x_1,x_2) = \int \mathrm{d}x'\; A(x_1,x')B(x',x_2)$$

We find (see Section ?? at the end of the document)

> MOYAL PRODUCT
>
> $$C(x,p) = A(x,p)\exp\left[\frac{i}{2}\left(\overleftarrow{\partial}_x\overrightarrow{\partial}_p - \overleftarrow{\partial}_p\overrightarrow{\partial}_x\right)\right]B(x,p)$$

remember x and p are four-vectors***. Let us suppose that the functions A and B vary in x and p with characteristic scales δx and δp respectively, then it's sufficient to truncate the exponential after the first term, that is

$$C(x,p) \approx AB + \frac{i}{2}(\partial_x A \partial_p B - \partial_p A \partial_x B) + \mathcal{O}(\delta x^2 \delta p^2)$$

This is the so-called *Gradient Expansion*, which can be written in terms of Poisson-Brackets:

$$C(x,p) \approx A(x,p)B(x,p) + \frac{i}{2}\{A,B\}_{\text{PB}}$$

note that this implies that the commutator becomes

$$[A\,[1pt]\circ, B] \xrightarrow{\text{Wigner}-\text{Transformation}} \approx i\{A,B\}_{\text{PB}}$$

Now, let us use the Gradient Expansion on the kinetic terms from the quantum kinetic equation:

$$\begin{aligned}[i\partial_t\,[1pt]\circ, F] &\to i\partial_t F \\ [-\partial_t^2\,[1pt]\circ, F] &\to 2i\omega\partial_t F \\ [-\mathcal{L}\{\}\,\boldsymbol{r}\,[1pt]\circ, F] &\to -2i\boldsymbol{k}\cdot\nabla_{\boldsymbol{r}} F \\ [\omega_{\boldsymbol{k}}^2\,[1pt]\circ, F] &\to -2i\omega_{\boldsymbol{k}}\boldsymbol{v}_{\boldsymbol{k}}\cdot\nabla_{\boldsymbol{r}} F\end{aligned}$$

on the other hand the collision integral becomes

$$\Sigma^{\text{K}} - F(\Sigma^{\text{R}} - \Sigma^{\text{A}}) - i(\partial_x \operatorname{Re}\Sigma^{\text{R}})\partial_p F + i(\partial_p \operatorname{Re}\Sigma^{\text{R}})\partial_x F$$

now combining it all and moving things around

$$\left[(1 - \partial_\omega \operatorname{Re}\Sigma^{\text{R}})\partial_t + (\partial_t \tilde{V})\partial_\omega + \tilde{\boldsymbol{v}}_{\boldsymbol{k}}\cdot\nabla_{\boldsymbol{r}} - (\nabla_{\boldsymbol{r}}\tilde{V})\cdot\nabla_{\boldsymbol{k}}\right] F = \mathcal{I}_{\text{coll}}[F]$$

where

$$\begin{aligned}\tilde{V} &= V(x) + \operatorname{Re}\Sigma^{\text{R}}(x,p) \\ \tilde{v}_{\boldsymbol{k}} &= \nabla_{\boldsymbol{k}}\left[\omega_{\boldsymbol{k}} + \operatorname{Re}\Sigma^{\text{R}}(x,p)\right] \\ \mathcal{I}_{\text{coll}}[F] &= i\Sigma^{\text{K}} + 2F\operatorname{Im}\Sigma^{\text{R}}\end{aligned}$$

Changing variables in F such that $\omega \to \omega - \omega_{\boldsymbol{k}} - \tilde{V}$ and using††† that $\omega = \varepsilon_{\boldsymbol{k}} + \tilde{V} + \operatorname{Re}\Sigma^{\text{R}}(\omega)$, which we solve for ω we obtain

$$\left[\tilde{Z}^{-1}\partial_t + \tilde{\boldsymbol{v}}_{\boldsymbol{k}}\cdot\nabla_{\boldsymbol{r}} - (\nabla_{\boldsymbol{r}}\tilde{V})\cdot\nabla_{\boldsymbol{k}}\right] F = \mathcal{I}_{\text{coll}}[F]$$

which looks a lot like the semi-classical Boltzmann Equation.

*** $\partial_x\partial_p = \nabla_{\boldsymbol{r}}\nabla_{\boldsymbol{k}} - \partial_t\partial_\varepsilon$

††† Jens writes "*Since F came from $G^{\text{K}} = (G^{\text{R}} - G^{\text{A}})F + \cdots$, having well-defined quasiparticles, i.e. a peaked spectral function*"

CONTENTS

VI. MOYAL PRODUCT

We start with

$$C(x_1, x_2) \equiv \int \mathrm{d}x_3 \; A(x_1, x_3) B(x_3, x_2)$$

and we are interested in the Wigner-Tranformation thereof. The Wigner-Transformation of a function $A(x_1, x_2)$ is:

$$A(x, p) \equiv \int \mathrm{d}x' \; e^{-ipx'} A(x + x'/2, x - x'/2)$$

and the inverse is

$$A(x_1, x_2) = \sum_p e^{ip(x_1 - x_2)} A\left[\frac{x_1 + x_2}{2}, p\right]$$

Let us plug in our Wigner-Transformation of A and B into the defintion of C:

$$C(x_1, x_2) \equiv \int \mathrm{d}x_3 \sum_{p_1} e^{ip_1(x_1 - x_3)} A\left[\frac{x_1 + x_3}{2}, p_1\right] \sum_{p_2} e^{ip'(x_3 - x_2)} B \Bigg[$$

If we once again define $x_1 = x + x'/2$ and $x_2 = x - x'/2$.

$$C(x + x'/2, x - x'/2) = \int \mathrm{d}x_3 \sum_{p_1} e^{ip_1(x + x'/2 - x_3)} A\left[\frac{x + x'/2 + x_3}{2}\right.$$

Thus it must be the case that

$$C(x, p) = \int \mathrm{d}x' \; e^{-ip'x'} \int \mathrm{d}x_3 \sum_{p_1} e^{ip_1(x + x'/2 - x_3)} A\left[\frac{x + x'/2 + x_3}{2},\right.$$

changing variables once again

$$x_A \equiv x_3 - x + x'/2, \qquad x_B \equiv x_3 - x - x'/2, \quad p_A = p_1 - p, \quad p_B =$$

giving us

$$C(x, p) = \iint \mathrm{d}x_A \mathrm{d}x_B \sum_{p_A, p_B} e^{i(p_B x_A - p_A x_B)} A[x + x_A/2, p + p_A] B[x -$$

expanding A and B in terms of p_A and p_B:

$$A(x, p + p_A) = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{\partial}{\partial p} A(x, p) p_A^n$$

So the integral

$$\sum_{p_A} e^{-ip_A x_B} p_A^n = (i)^n \delta^{(n)}(x_B), \qquad \sum_{p_B} e^{ip_B x_A} p_B^n = (-i)^n \delta^{(n)}(x_A)$$

However, in order to evaluate the n^{th} derivative of the delta-function we need to differentiate the rest of the integrand w.r.t the delta-function's argument (partial integration). So expanding A gives us a n^{th} derivative of A w.r.t p_A and an n^{th} derivative of B w.r.t x_B (the chain rule gives us factor 2 each time we take an x-derivative). So we can write this as

$$C(x,p) = \left[\sum_{mn} \frac{1}{n!m!}\left(\frac{i}{2}\partial_{p_A}\partial_{x_B}\right)^n \left(-\frac{i}{2}\partial_{p_B}\partial_{x_A}\right)^m A(x_A,p_A)B(x_B,p_B)\right]_{x_A=x_B=x}^{p_A=p_B=p}$$

which we can write as

$$C(x,p) = A(x,p)\exp\left[\frac{i}{2}\left(\overleftarrow{\partial}_p\vec{\partial}_x - \overleftarrow{\partial}_x\vec{\partial}_p\right)\right]B(x,p)$$