EM1 AND EM2 IN FIVE EQUATIONS:

$$\nabla \times \boldsymbol{E} = -\frac{\partial \mathbf{B}}{\partial t}$$
$$\nabla \cdot \boldsymbol{E} = \frac{\rho}{\varepsilon_0}$$
$$\nabla \times \boldsymbol{B} = \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$
$$\nabla \cdot \boldsymbol{B} = 0$$
$$\mathbf{F} = q\left(\boldsymbol{E} + \mathbf{v} \times \boldsymbol{B}\right)$$

Which can also be expressed with respect to the potentials

$$\nabla^2 V + \frac{\partial}{\partial t} \nabla \cdot \mathbf{A} = -\frac{\rho}{\varepsilon_0}$$
$$\nabla^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} - \nabla\left(\nabla \cdot \mathbf{A} + \frac{1}{c^2}\frac{\partial V}{\partial t}\right) = -\mu_0 \mathbf{J}$$
$$\mathbf{F} = q\left(-\nabla V - \frac{\partial \mathbf{A}}{\partial t} + \mathbf{v} \times (\nabla \times \mathbf{A})\right)$$

POTENTIALS

A. Scalar Potential

$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_{\mathcal{V}} \frac{\rho(\mathbf{r}')}{\mathscr{r}} \mathrm{d}\tau' = \frac{1}{4\pi\varepsilon_0}\int_{\mathcal{A}} \frac{\sigma(\mathbf{r}')}{\mathscr{r}} \mathrm{d}a' = \frac{1}{4\pi\varepsilon_0}\int_{\mathcal{L}} \frac{\lambda(\mathbf{r}')}{\mathscr{r}} \mathrm{d}\ell'$$

Vektorpotentialet

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_{\mathcal{V}} \frac{\mathbf{J}(\mathbf{r}')}{\mathscr{r}} \mathrm{d}\tau' = \frac{\mu_0}{4\pi}\int_{\mathcal{A}} \frac{\mathbf{K}(\mathbf{r}')}{\mathscr{r}} \mathrm{d}a' = \frac{\mu_0}{4\pi}\int_{\mathcal{L}} \frac{\mathbf{I}(\mathbf{r}')}{\mathscr{r}} \mathrm{d}\ell'$$

where $\mathcal{V}$, $\mathcal{A}$ and $\mathcal{L}$ are the volume, area and line one integrates over

B. Retarded Potentials

When we calculated the retarded potentials we use the same formulae as above, but we use the charges and currents at the retarded time, $t_R \equiv t - \mathscr{r}/c$.

C. Gauge Freedom

$$\mathbf{A}' = \mathbf{A} + \nabla\lambda$$
$$V' = V - \frac{\partial \lambda}{\partial t}$$

CIRCUITS

D. Impedances

The real part of the impedance is the resistance, R, and the imaginary part is the reactance, X:

$$Z = R + iX$$

Impedances

Resistor:	$Z = R$
Inductor:	$Z = i\omega L$
Capacitor:	$Z = \frac{1}{i\omega C} = -\frac{i}{\omega C}$

Characteristic times and frequencies

RC-circuit	$\tau = RC$
LR-circuit	$\tau = \frac{L}{R}$
LC-circuit	$\omega_0 = \frac{1}{\sqrt{LC}}$

When we connect two impedances in *series* we have that

$$Z_{eff} = Z_1 + Z_2$$

whereas if they are connected in *parallel* we get

$$Z_{eff} = \frac{Z_1 Z_2}{Z_1 + Z_2} = \frac{1}{Z_1^{-1} + Z_2^{-1}}$$

If $Z, Z_1, Z_2 \in \mathbb{C}$ then it is the case that

$$\left|\frac{1}{Z}\right| = \frac{1}{|Z|}$$
$$|Z_1 Z_2| = |Z_1||Z_2|$$

sometimes it is beneficial to write

$$Z = |Z|e^{i\phi}, \quad \phi = \arctan\left(\frac{\mathrm{Im}(Z)}{\mathrm{Re}(Z)}\right)$$

Ohm's Law for complex alternating currents

$$\tilde{I} = \frac{\tilde{\mathcal{E}}}{Z}$$

Let $\tilde{\mathcal{E}}(t) = \tilde{\mathcal{E}}_0 e^{i\omega t}$, then

$$\tilde{I}(t) = \frac{\tilde{\mathcal{E}}_0}{|Z|} e^{i(\omega t - \phi)}, \quad \phi = \arctan\left(\frac{\mathrm{Im}(Z)}{\mathrm{Re}(Z)}\right)$$

Power:

$$P(t) \equiv \mathcal{E}(t) I(t) = \frac{\mathcal{E}_0^2}{|Z|} \cos(\omega t)\cos(\omega t - \phi)$$
$$\langle P \rangle = \frac{\mathcal{E}_0^2}{2|Z|}\cos(\phi) = \frac{\mathcal{E}_0^2 R}{2|Z|^2}$$

Energy:

In a capacitor

$$W_C = \frac{Q^2}{2C}$$

In an inductor

$$W_L = \frac{LI^2}{2}$$

E. Kirchhoff's Laws

1. The current entering a node is equal to the current leaving the node

2. $\mathcal{E}$ is equal to the voltage difference over the entire circuit

It's straightforward to calculate the emf when the components are in series, but it can be quite a challenge when they are in parallel

These laws also hold for subcircuits.

Changing B-fields induce an emf:

$$\Phi = \int \mathbf{B} \cdot \mathrm{d}\mathbf{a}$$
$$\mathcal{E} = -\frac{\mathrm{d}\Phi}{\mathrm{d}t}$$

ENERGY, MOMENTUM, FORCE ETC.

Poynting Vector:

$$\mathbf{S} = \frac{1}{\mu_0} \boldsymbol{E} \times \boldsymbol{B}$$

$\mathbf{S}$ is the energy-flux-density – how much energy that flows through an infinitesimal area per unit time **Intensity**

$$I \equiv \langle S \rangle = \frac{1}{2\mu_0} EB$$

Radiation Pressure

$$P = \sqrt{\mu_0 \varepsilon_0}\langle S \rangle = \frac{I}{c} = \frac{1}{2}\varepsilon_0 E_0^2$$

which is multiplied by two if the light is reflected

Poynting's Theorem:

$$\frac{\mathrm{d}W}{\mathrm{d}t} = -\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{V}} u \,\mathrm{d}\tau - \oint_{\mathcal{S}} \mathbf{S} \cdot \mathrm{d}\mathbf{a}$$

if no work is exerted on the system then

$$\frac{\partial u}{\partial t} = -\nabla \cdot \mathbf{S}$$

the change in energy is minus the divergence of the Poynting vector

Electromagnetic energy-density:

$$u = \frac{1}{2}\left(\varepsilon_0 E^2 + \frac{1}{\mu_0} B^2\right)$$

Total energy

$$U = \int_{\mathcal{V}} u \,\mathrm{d}\tau$$

Continuity Equation

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot \mathbf{J}$$

Maxwell's Stress Tensor:

$$\overleftrightarrow{\mathbf{T}}_{ij} \equiv \varepsilon_0 \left(E_i E_j - \frac{1}{2}\delta_{ij} E^2\right) + \frac{1}{\mu_0}\left(B_i B_j - \frac{1}{2}\delta_{ij} B^2\right)$$

Electromagnetic Forces

$$\mathbf{F} = \oint_{\mathcal{S}} \overleftrightarrow{\mathbf{T}} \cdot \mathrm{d}\mathbf{a} - \mu_0 \varepsilon_0 \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{V}} \mathbf{S} \,\mathrm{d}\tau$$

$\overleftrightarrow{\mathbf{T}}$ is the force/stress per unit area that is exerted on the surface $\mathcal{S}$. T_{xx}, T_{yy} and T_{zz} are pressures that are respectively exerted in the x, y and z directions, T_{ij} where $i \neq j$ are stresses, so forces per area that are exerted orthogonal to the normal of ij-plane.

$\overleftrightarrow{\mathbf{T}}$ also describes the *momentum flux.*

Momentum Density

$$\mathbf{g} = \mu_0 \varepsilon_0 \mathbf{S}$$

The momentum density can be integrated over a volume, $\mathcal{V}$, and you obtain the momentum

$$\mathbf{p} = \int_{\mathcal{V}} \mathbf{g} \mathrm{d}\tau$$

There is also a momentum-balance equation:

$$\mathbf{f} = \nabla \cdot \overleftrightarrow{\mathbf{T}} - \frac{\partial \mathbf{g}}{\partial t}$$

Angular Momentum Density

$$\boldsymbol{\ell} = \mathbf{r} \times \mathbf{g}$$

Electromagnetic Waves

$$\tilde{\boldsymbol{B}}_0 = \frac{1}{c} \hat{\mathbf{k}} \times \tilde{\mathbf{E}}_0$$

EM waves in a conductor:

now $\mathbf{k} \in \mathbb{C}$:

$$\tilde{\mathbf{k}} = (k + i\kappa)\hat{\mathbf{k}}$$

where

$$k \equiv \omega \sqrt{\frac{\varepsilon\mu}{2}} \left(\sqrt{1 + \left(\frac{\sigma}{\varepsilon\omega}\right)^2} + 1 \right)^{\frac{1}{2}}$$

$$\kappa \equiv \omega \sqrt{\frac{\varepsilon\mu}{2}} \left(\sqrt{1 + \left(\frac{\sigma}{\varepsilon\omega}\right)^2} - 1 \right)^{\frac{1}{2}}$$

Therefore the amplitude attenuates

$$\tilde{\mathbf{E}}(z,t) = E_0 e^{-\kappa z} e^{i(kz - \omega t + \delta)} \hat{\boldsymbol{x}}$$

$$\tilde{\mathbf{B}}(z,t) = \frac{K E_0}{\omega} e^{-\kappa z} e^{i(kz - \omega t + \delta + \phi)} \hat{\boldsymbol{y}}$$

where I have assumed that $\mathbf{k} \parallel \hat{\boldsymbol{z}}$ and that the light is polarised along $\hat{\boldsymbol{x}}$. K is the magnitude of $\tilde{k}$:

$$K = \sqrt{k^2 + \kappa^2}$$

and

$$\phi = \arctan\left(\frac{\kappa}{k}\right)$$

The relation between the magnitudes of E and B are

$$\frac{B_0}{E_0} = \frac{K}{\omega} = \sqrt{\varepsilon\mu \sqrt{1 + \left(\frac{\sigma}{\varepsilon\omega}\right)^2}}$$

The following still holds

$$\lambda = \frac{2\pi}{k}, \quad v = \frac{\omega}{k}, \quad n = \frac{ck}{\omega}$$

but the Poynting vector changes a bit because we no longer use the vacuum permeability, $\mu = \mu_r \mu_0$:

$$\mathbf{S} = \frac{1}{\mu} \boldsymbol{E} \times \boldsymbol{B}$$

Continuity of the Fields at Interfaces

$$D_2^{\perp} - D_1^{\perp} = \sigma_f$$

$$B_2^{\perp} - B_1^{\perp} = 0$$

$$\boldsymbol{E}_2^{\parallel} - \boldsymbol{E}_1^{\parallel} = \mathbf{0}$$

$$\mathbf{H}_2^{\parallel} - \mathbf{H}_1^{\parallel} = \mathbf{K}_f \times \hat{\mathbf{n}}$$

Auxiliary fields:

The auxiliary fields are defined as

$$\mathbf{D} = \varepsilon_0 \boldsymbol{E} + \mathbf{P}$$

$$\mathbf{H} = \frac{1}{\mu_0} \boldsymbol{B} - \mathbf{M}$$

in linear media the susceptibility is a scalar:

$$\mathbf{P} = \varepsilon_0 \chi_e \boldsymbol{E}$$

$$\mathbf{M} = \chi_m \mathbf{H}$$

and hence

$$\mathbf{D} = \varepsilon \boldsymbol{E}$$

$$\mathbf{H} = \frac{1}{\mu} \boldsymbol{B}$$

where

$$\varepsilon = \varepsilon_0 (1 + \chi_e) = \varepsilon_0 \varepsilon_r$$

$$\mu = \mu_0 (1 + \chi_m) = \mu_0 \mu_r$$

OPTICS

Snell's Law

$$\frac{\sin\theta_T}{\sin\theta_I} = \frac{n_1}{n_2}$$

Refractive Index

$$n \equiv \sqrt{\mu_r \varepsilon_r} \approx \sqrt{\varepsilon_r}$$

the approximation holds because for many materials $\mu_r \approx 1$.

Coefficients of reflection and transmission

$$\tilde{E}_{0,R} = \left(\frac{\alpha - \beta}{\alpha + \beta}\right) \tilde{E}_{0,I}$$
$$\tilde{E}_{0,T} = \left(\frac{2}{\alpha + \beta}\right) \tilde{E}_{0,I}$$

where

$$\alpha \equiv \frac{\cos\theta_T}{\cos\theta_I}$$
$$\beta \equiv \frac{\mu_1 v_1}{\mu_2 v_2} = \frac{\mu_1 n_2}{\mu_2 n_1} \approx \frac{v_1}{v_2}$$

We can define the coefficients as

$$R \equiv \frac{I_R}{I_I} = \left(\frac{E_{0,R}}{E_{0,I}}\right)^2 = \left(\frac{\alpha - \beta}{\alpha + \beta}\right)^2$$
$$T \equiv \frac{I_T}{I_I} = \alpha \frac{\varepsilon_2 v_2}{\varepsilon_1 v_1} \left(\frac{E_{0,T}}{E_{0,I}}\right)^2 = \alpha\beta \left(\frac{2}{\alpha + \beta}\right)^2$$

IMPORTANT FORMULAE

The electric field of a charged particle that is accelerating arbitrarily

$$\mathbf{E}(\mathbf{r}, t) = \frac{q}{4\pi\varepsilon_0} \frac{\mathscr{r}'}{(\boldsymbol{\mathscr{r}}' \cdot \mathbf{u}')^3} \left((c^2 - v'^2)\mathbf{u}' + \boldsymbol{\mathscr{r}}' \times (\mathbf{u}' \times \mathbf{a}')\right)$$

the $'$ is there to remind us to use the retarded values. The magnetic field is:

$$\boldsymbol{B}(\mathbf{r}, t) = \frac{1}{c} \hat{\boldsymbol{\mathscr{r}}}' \times \boldsymbol{E}(\mathbf{r}, t)$$

This simplifies if the velocity is constant:

$$\mathbf{E}(\mathbf{r}, t) = \frac{q}{4\pi\varepsilon_0} \frac{1 - v^2/c^2}{\left(1 - v^2 \sin^2\theta/c^2\right)^{3/2}} \frac{\hat{\mathbf{R}}}{R^2}$$

$\mathbf{R}$ points from the *current* position of the charged particle to where we measure the field. The magnetic field:

$$\boldsymbol{B}(\mathbf{r}, t) = \frac{1}{c^2} \mathbf{v} \times \boldsymbol{E}$$

Electric Dipole Radiation:

Assume the dipole moment is

$$\mathbf{p}(t) = q_0 \cos(\omega t)\mathbf{d} = \mathbf{p}_0 \cos(\omega t)$$

and that $\mathbf{p} \parallel \hat{\boldsymbol{z}}$.

Then it holds that

$$V(\mathbf{r}, t) = \frac{q_0}{4\pi\varepsilon_0} \left(\frac{\cos(\omega(t - \mathscr{r}_+/c))}{\mathscr{r}_+} - \frac{\cos(\omega(t - \mathscr{r}_-/c))}{\mathscr{r}_-}\right)$$

where

$$\mathscr{r}_\pm = \sqrt{r^2 \mp rd\cos\theta + (d/2)^2}$$

and θ is the angle from the z-axis.

To calculate the fields we must make three assumptions

1. $d \ll r$
2. $d \ll \frac{c}{\omega}$
3. $r \gg \frac{c}{\omega}$

because then the scalar potential simplifies to

$$V(r, \theta, t) = -\frac{p_0\omega}{4\pi\varepsilon_0 c} \left(\frac{\cos\theta}{r}\right) \sin\left(\omega(t - r/c)\right)$$

and the vector potential becomes

$$\mathbf{A}(r, \theta, t) = -\frac{\mu_0 p_0 \omega}{4\pi r} \sin(\omega(t - r/c))\hat{\boldsymbol{z}}$$

The fields:

$$\boldsymbol{E}(r, \theta, t) = -\frac{\mu_0 p_0 \omega^2}{4\pi} \left(\frac{\sin\theta}{r}\right) \cos(\omega(t - r/c))\hat{\boldsymbol{\theta}}$$
$$\boldsymbol{B}(r, \theta, t) = -\frac{\mu_0 p_0 \omega^2}{4\pi c} \left(\frac{\sin\theta}{r}\right) \cos(\omega(t - r/c))\hat{\boldsymbol{\varphi}}$$

Poynting vector:

$$\mathbf{S} = \frac{\mu_0}{c} \left(\frac{p_0\omega^2}{4\pi} \left(\frac{\sin\theta}{r}\right) \cos(\omega(t - r/c))\right)^2 \hat{\boldsymbol{r}}$$

Intensity:

$$I = \langle S \rangle = \left(\frac{\mu_0 p_0^2 \omega^4}{32\pi^2 c}\right) \frac{\sin^2\theta}{r^2}$$

Power

$$\langle P \rangle = \frac{\mu_0 p_0^2 \omega^4}{12\pi c}$$

Magnetic Dipole Radiation:

Generally the magnetic dipole moment is given by

$$\mathbf{m} = I \int \mathrm{d}\mathbf{a}$$

Assume now that $I = I_0 \cos(\omega t)$, then

$$\mathbf{m}(t) = \mathbf{m}_0 \cos(\omega t)$$

If $\mathbf{m} \parallel \hat{\boldsymbol{z}}$, then it is reasonable to assume that the current runs in along circular path with radius b:

$$m_0 = \pi b^2 I_0$$

We need to assume

1. $b \ll r$
2. $b \ll \frac{c}{\omega}$
3. $r \gg \frac{c}{\omega}$

Then we obtain the vector potential

$$\mathbf{A}(r,\theta,t) = -\frac{\mu_0 m_0 \omega}{4\pi c}\left(\frac{\sin\theta}{r}\right)\sin(\omega(t-r/c))\hat{\boldsymbol{\varphi}}$$

Fields:

$$\mathbf{E}(r,\theta,t) = \frac{\mu_0 m_0 \omega^2}{4\pi c}\left(\frac{\sin\theta}{r}\right)\cos(\omega(t-r/c))\hat{\boldsymbol{\varphi}}$$
$$\mathbf{B}(r,\theta,t) = -\frac{\mu_0 m_0 \omega^2}{4\pi c^2}\left(\frac{\sin\theta}{r}\right)\cos(\omega(t-r/c))\hat{\boldsymbol{\theta}}$$

Poynting vector:

$$\mathbf{S} = \frac{\mu_0}{c}\left(\frac{m_0\omega^2}{4\pi c}\left(\frac{\sin\theta}{r}\right)\cos(\omega(t-r/c))\right)^2\hat{\boldsymbol{r}}$$

Intensity

$$I = \langle S\rangle = \left(\frac{\mu_0 m_0^2 \omega^4}{32\pi^2 c^3}\right)\frac{\sin^2\theta}{r^2}$$

Power

$$\langle P\rangle = \frac{\mu_0 m_0^2 \omega^4}{12\pi c^3}$$

RELATIVITY

Lorentz transformation:

$$(x^\mu)' = \Lambda^\mu_\nu x^\nu$$

Assuming the particle is moving in the x^1 direction then:

$$\Lambda = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where

$$(x^0, x^1, x^2, x^3) \equiv (ct, x, y, z)$$
$$\beta \equiv \frac{v}{c}$$
$$\gamma \equiv \frac{1}{\sqrt{1-\beta^2}}$$

The inverse Lorentz transformation

$$\Lambda^{-1} = \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

assuming the particle is moving in the x-direction we can also transform the fields:

$$E'_x = E_x$$
$$E'_y = \gamma(E_y - vB_z)$$
$$E'_z = \gamma(E_z + vB_y)$$
$$B'_x = B_x$$
$$B'_y = \gamma(B_y + \frac{v}{c^2}E_z)$$
$$B'_z = \gamma(B_z - \frac{v}{c^2}E_y)$$

We can define the electromagnetic field-tensor

$$F^{\mu\nu} = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{pmatrix}$$

and the dual tensor

$$G^{\mu\nu} = \begin{pmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & -E_z/c & E_y/c \\ -B_y & E_z/c & 0 & -E_x/c \\ -B_z & -E_y/c & E_x/c & 0 \end{pmatrix}$$

and finally the current density four-vector

$$J^\mu = (c\rho, J_x, J_y, J_z)$$

then the continuity equation, Equation E, becomes:

$$\frac{\partial J^\mu}{\partial x^\mu} = 0$$

Maxwell's Equations are reduced to

$$\frac{\partial F^{\mu\nu}}{\partial x^\nu} = \mu_0 J^\mu, \quad \frac{\partial G^{\mu\nu}}{\partial x^\nu} = 0$$

And the four-potential is

$$A^\mu = (V/c, A_x, A_y, A_z)$$

which means that

$$F^{\mu\nu} = \frac{\partial A^\nu}{\partial x_\mu} - \frac{\partial A^\mu}{\partial x_\nu}$$

MATHEMATICS

The nightmare Equation

$$((\mathbf{A}\cdot\nabla)\mathbf{B})_j = A_i\partial_i B_j$$

then

$$(\mathbf{A}\cdot\nabla)\mathbf{B} = A_i\partial_i B_j\mathbf{e}_j$$

Cross product

$$\mathbf{A}\times\mathbf{B} = \mathbf{e}_i\varepsilon_{ijk}A_jB_k$$

F. Integration

Gradient theorem:

$$\int_{\mathbf{a}}^{\mathbf{b}} \nabla f \mathrm{d}\boldsymbol{\ell} = f(\mathbf{b}) - f(\mathbf{a})$$

Divergence Theorem (Gauss'):

$$\int (\nabla\cdot\mathbf{A})\mathrm{d}\tau = \oiint \mathbf{A}\cdot\mathrm{d}\mathbf{a}$$

Curl theorem (Stokes'):

$$\int (\nabla\times\mathbf{A})\cdot\mathrm{d}\mathbf{a} = \oint \mathbf{A}\cdot\mathrm{d}\boldsymbol{\ell}$$

Gradient of $\mathscr{r}^{-1}$:

$$\nabla'\left(\frac{1}{\mathscr{r}}\right) = \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} = -\nabla\left(\frac{1}{\mathscr{r}}\right)$$

G. Vector derivatives

Infinitesimals

Cartesian:

$$\mathrm{d}\boldsymbol{\ell} = \mathrm{d}x\hat{\boldsymbol{x}} + \mathrm{d}y\hat{\boldsymbol{y}} + \mathrm{d}z\hat{\boldsymbol{z}}$$
$$\mathrm{d}\tau = \mathrm{d}x\mathrm{d}y\mathrm{d}z$$

Spherical:

$$\mathrm{d}\boldsymbol{\ell} = \mathrm{d}r\,\hat{\boldsymbol{r}} + r\mathrm{d}\theta\hat{\boldsymbol{\theta}} + r\sin\theta\mathrm{d}\varphi\hat{\boldsymbol{\varphi}}$$
$$\mathrm{d}\tau = r^2\sin\theta\mathrm{d}r\mathrm{d}\theta\mathrm{d}\varphi$$

Cylindrical:

$$\mathrm{d}\boldsymbol{\ell} = \mathrm{d}s\hat{\mathbf{s}} + s\mathrm{d}\theta\hat{\boldsymbol{\theta}} + \mathrm{d}z\hat{\boldsymbol{z}}$$
$$\mathrm{d}\tau = s\mathrm{d}s\mathrm{d}\theta\mathrm{d}z$$

Gradienten

Cartesian:

$$\nabla\beta = \frac{\partial\beta}{\partial x}\hat{\mathbf{x}} + \frac{\partial\beta}{\partial y}\hat{\mathbf{y}} + \frac{\partial\beta}{\partial z}\hat{\mathbf{z}}$$

Cylindrical:

$$\nabla\beta = \frac{\partial\beta}{\partial s}\hat{\mathbf{s}} + \frac{1}{s}\frac{\partial\beta}{\partial\theta}\hat{\boldsymbol{\theta}} + \frac{\partial\beta}{\partial z}\hat{\mathbf{z}}$$

Spherical

$$\nabla\beta = \frac{\partial\beta}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial\beta}{\partial\theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial\beta}{\partial\varphi}\hat{\boldsymbol{\varphi}}$$

the gradient of a vector

$$\vec{v}\cdot\nabla\vec{v} = v_x\frac{\partial\vec{v}}{\partial x} + v_y\frac{\partial\vec{v}}{\partial y} + v_z\frac{\partial\vec{v}}{\partial z}$$

Divergence

Cartesian:

$$\nabla\cdot\vec{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

Cylindrical:

$$\nabla\cdot\vec{v} = \frac{1}{s}\frac{\partial}{\partial s}(sv_s) + \frac{1}{s}\frac{\partial v_\theta}{\partial\theta} + \frac{\partial v_z}{\partial z}$$

Spherical

$$\nabla\cdot\vec{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(v_\theta\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\varphi}{\partial\varphi}$$

Curl

Cartesian:

$$\nabla\times\vec{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{\mathbf{z}}$$

Spherical:

$$\nabla\times\vec{v} = \frac{1}{r\sin\theta}\left(\frac{\partial}{\partial\theta}(v_\varphi\sin\theta) - \frac{\partial v_\theta}{\partial\varphi}\right)\hat{\mathbf{r}} + \frac{1}{r}\left(\frac{\partial}{\partial r}(rv_\theta) - \frac{\partial v_r}{\partial\theta}\right)\hat{\boldsymbol{\varphi}} + \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial v_r}{\partial\varphi} - \frac{\partial}{\partial r}(rv_\varphi)\right)\hat{\boldsymbol{\theta}}$$

Cylindrical:

$$\nabla \times \vec{v} = \frac{1}{s} \begin{vmatrix} \hat{\mathbf{s}} & s\hat{\boldsymbol{\varphi}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial s} & \frac{\partial}{\partial \varphi} & \frac{\partial}{\partial z} \\ v_s & s v_\varphi & v_z \end{vmatrix}$$

Laplacian

Cartesian:

$$\nabla^2 \beta = \frac{\partial^2 \beta}{\partial x^2} + \frac{\partial^2 \beta}{\partial y^2} + \frac{\partial^2 \beta}{\partial z^2}$$

Spherical:

$$\nabla^2 u = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial u}{\partial r} \right) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial u}{\partial \theta} \right) + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2 u}{\partial \varphi^2}$$

Cylindrical:

$$\nabla^2 u = \frac{1}{s} \frac{\partial}{\partial s} \left(s \frac{\partial u}{\partial s} \right) + \frac{1}{s^2} \frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2}$$