I. QUANTISATION OF THE ELECTROMAGNETIC FIELD

The ultraviolet catastrophe as well as the photoelectric effect motivated physicists to try and rethink the theory of electromagnetism. Physicists realised that the Fourier decomposition of a vector potential that satisfies Maxwell's equations:

$$\boldsymbol{A}(\boldsymbol{x},t) = \sum_\lambda \int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3}\left(\boldsymbol{a}_{\boldsymbol{k},\lambda}e^{i\boldsymbol{k}\cdot\boldsymbol{x}} + \boldsymbol{a}^*_{\boldsymbol{k},\lambda}e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}\right)$$

can be expressed as a Harmonic Oscillator, by introducing the following variables:

$$\begin{aligned} \boldsymbol{Q}_{\boldsymbol{k}} &= \boldsymbol{a}_{\boldsymbol{k}\lambda} + \boldsymbol{a}^*_{\boldsymbol{k}\lambda} \quad \rightsquigarrow \quad \dot{\boldsymbol{Q}}_{\boldsymbol{k}} = \boldsymbol{P}_{\boldsymbol{k}} \\ \boldsymbol{P}_{\boldsymbol{k}} &= -i\omega_{\boldsymbol{k}}\left(\boldsymbol{a}_{\boldsymbol{k}\lambda} - \boldsymbol{a}^*_{\boldsymbol{k}\lambda}\right) \quad \rightsquigarrow \quad \dot{\boldsymbol{Q}}_{\boldsymbol{k}} = -\omega_{\boldsymbol{k}^2}\boldsymbol{Q}_{\boldsymbol{k}} \end{aligned}$$

where we've used that $\dot{\boldsymbol{a}_{\boldsymbol{k}}} = i\omega_{\boldsymbol{k}}\boldsymbol{a}_{\boldsymbol{k}}$. This means that the coefficients in the vector potential above can be expressed as a Harmonic Oscillator Hamiltonian:

$$H = \frac{1}{2}\int \frac{\mathrm{d}^3\boldsymbol{k}_{\boldsymbol{n}}}{(2\pi)^3}\left(\boldsymbol{P}^2_{\boldsymbol{k}} + \omega^2_{\boldsymbol{k}}\boldsymbol{Q}^2_{\boldsymbol{k}}\right)$$

which in fact can be written as

$$H = \frac{1}{2}\int \mathrm{d}^3\boldsymbol{x}\left(\boldsymbol{E}^2 + \boldsymbol{B}^2\right)$$

which we know to be the total energy of the electromagnetic field. We know how to quantise Harmonic Oscillators, we transform from Poisson Brackets to commutators:

$$[\hat{\boldsymbol{Q}}_{\boldsymbol{k}}, \hat{\boldsymbol{P}}_{\boldsymbol{k}'}] = i\hbar(2\pi)^3\delta^{(3)}(\boldsymbol{k}-\boldsymbol{k}')$$

and the Hamiltonian becomes an operator. As we did for the regular harmonic oscillator, this can be expressed in terms of bosonic creation and annihilation operators:

$$\hat{H} = \int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3}\hbar\omega_{\boldsymbol{k}}\sum_{\lambda=1,2}\left(\hat{\boldsymbol{a}}^\dagger_{\boldsymbol{k}\lambda}\hat{\boldsymbol{a}}_{\boldsymbol{k}\lambda} + \frac{1}{2}\right)$$

whence we can define the vector potential operator

$$\hat{\boldsymbol{A}}(\boldsymbol{x},t) = \int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3\sqrt{2\omega_{\boldsymbol{k}}}}\left(\hat{\boldsymbol{a}}_{\boldsymbol{k}\lambda}(t)e^{i\boldsymbol{k}\cdot\boldsymbol{x}} + \hat{\boldsymbol{a}}^\dagger_{\boldsymbol{k}\lambda}e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}\right)$$

Clearly the quantum of light, a photon, is a boson, therefore we describe the many body state by a bosonic Fock state, where occupation numbers can be any non-negative integer. The creation operators, $\hat{\boldsymbol{a}}^\dagger_{\boldsymbol{k}\lambda}$ creates a photon with wave-vector $\boldsymbol{k}$ and polarisation λ, whereas the annihilation operator annihilates the same photon.

II. DIRAC EQUATION

The Schrödinger Equation describes the dynamics of non-relativistic quantum mechanical particles, but at the time there was no way of describing relativistic quantum mechanical particles. The only equation one had that came close was the Klein-Gordon Equation

$$(\Box + m^2)\phi = 0, \qquad \Box = \eta^{\mu\nu}\partial_\mu\partial_\nu$$

However, the issue with the Klein-Gordon equation was not that it predicted negative energy states, but that superposition states that included both positive and negative energy states did not have probability conservation. This is an issue because the leading interpretation at the time was the probabilistic interpretation of quantum mechanics.

Dirac's trick to "take the square root of the Klein-Gordon Equation"was that he needed to use matrices as coefficients. Consider

$$\begin{aligned} &(\alpha_i p_i + \beta m)^2 = p_i^2 + m^2 \\ &= \beta^2 m^2 + \sum_i \alpha_i^2 p_i^2 + \{\alpha_i, \beta\}\, p_i m + \sum_j \{\alpha_i, \alpha_j\}\, p_i p_j \end{aligned}$$

hence we require that $\beta^2 = \nVdash$, $\{\alpha_i, \alpha_j\} = 2\delta_{ij}\nVdash$ and $\{\alpha_i, \beta\} = \{\alpha_i, \alpha_j\} = 0$. By multiplying by β from the left we obtain the γ-matrices:

$$\gamma^0 = \beta, \qquad \gamma^i = \beta\alpha^i$$

with these matrices we see that the matrices required for the Dirac Equation satisfy the Clifford Algebra:

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\nVdash$$

It can be shown that α_i, β as well as γ^μ must be 4×4 matrices if we require $(3+1)$-spacetime dimensions. Additionally we have that

$$\alpha^i = \sigma^x \otimes \sigma^i, \quad \beta = \sigma_z \otimes \nVdash$$

(and corresponding relations for the γ-matrices). The fact that the Pauli matrices appear is not a coincidence, in fact if we were to look at the non-relativistic limit of the Dirac equation the 4-component differential equation would split into two 2-component differential equations: the first one is the Pauli Equation, given below, and the second one is approximately zero:

$$H = \frac{1}{2}\left((\boldsymbol{p} - q\boldsymbol{A})^2 - q\hbar\boldsymbol{\sigma}\cdot\boldsymbol{B}\right) + q\phi$$

we see that the Dirac Equation *inherently* describes spin, and its interaction with the electromagnetic field. The Dirac equation is

$$H_{\text{Dirac}} = \boldsymbol{\alpha}\cdot\boldsymbol{p} + \beta m$$

but the Dirac Equation is often written as

$$(\gamma^\mu p_\mu - m)\psi = 0$$

A. Useful γ-matrix identities

The relation between the α and β-matrices and the γ-matrices is

$$\boxed{\gamma^0 = \beta, \qquad \gamma^i = \beta\alpha^i}$$

We often need to calculate traces of different combinations of γ-matrices

$$\begin{aligned}
\mathrm{Tr}\,[\gamma^\mu] &= 0\\
\mathrm{Tr}\,[\gamma_5] &= 0\\
\mathrm{Tr}\,[\gamma_5\gamma^\mu] &= 0\\
\mathrm{Tr}\,[\gamma^\mu\gamma^\nu] &= 4\eta^{\mu\nu}\\
\mathrm{Tr}\,[\gamma_5\gamma^\mu\gamma^\nu] &= 0\\
\mathrm{Tr}\,[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma] &= 4\left(\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho}\right)\\
\mathrm{Tr}\,[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_5] &= -4i\varepsilon^{\mu\nu\rho\sigma}\\
\overline{\gamma^\mu} &= \gamma^\mu
\end{aligned}$$

The “fifth"γ-matrix is defined as

$$\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$$

and it behaves very much like other γ-matrices; it squares to the identity and anti-commutes with the other γ-matrices. Due to the fact that γ_5 is an involutory matrix, it has eigenvalues ± 1. It turns out that these eigenvalues represent the *chirality* of the left and right spinors:

$$\gamma_5\psi_L = -\psi_L, \qquad \gamma_5\psi_R = \psi_R$$

All 4×4 matrices can be written as a linear combination of the following set of matrices $\{\mathbb{1}, \gamma^\mu, \gamma_5, \gamma^\mu\gamma_5, [\gamma^\mu, \gamma^\nu]\}$, *i.e*

$$M = a_0\mathbb{1} + a_\mu\gamma^\mu + a_5\gamma_5 + a_{\mu 5}\gamma^\mu\gamma_5 + a_{\mu\nu}[\gamma^\mu, \gamma^\nu]$$

We can find the coefficients by means of

$$\mathrm{Tr}[M] = 4a_0, \quad \mathrm{Tr}[M\gamma^\mu] = 4a_\mu, \quad \mathrm{Tr}[M\gamma_5] = 4a_4$$
$$\mathrm{Tr}[M\gamma^\nu\gamma_5] = -16a_{\nu 5}$$

and similar identities for the remaining matrices in the basis.

We will also have to use

$$\boxed{\frac{1}{2}\,\mathrm{Tr}\left[\gamma^0\gamma_\mu\gamma^0\gamma_\nu\right]p^\mu q^\nu = 2\left[E_pE_q + (\boldsymbol{p}\cdot\boldsymbol{q})\right]}$$

B. Dirac Conjugation

It turns out that the bilinear we are used to using $\psi^\dagger\psi$ is not Lorentz invariant, therefore we introduce the Dirac conjugate $\bar{\psi}$:

$$\bar{\psi} \equiv \psi^\dagger\gamma^0$$

Its main property is that

$$(U_{\frac{1}{2}}\psi)^\dagger\gamma^0 = \bar{\psi}U_{\frac{1}{2}}^{-1}$$

where we have defined the following unitary matrix

$$U_{\frac{1}{2}}(\omega) \equiv \exp\left(\frac{i}{2}\omega_{ij}S^{ij}\right)$$

Using the this new conjugation we can build scalars, vectors and tensors:

$$\begin{aligned}
\bar{\psi}\psi &: \text{ Scalar}\\
\bar{\psi}\gamma^\mu\psi &: \text{ Four-vector}\\
\bar{\psi}[\gamma^\mu, \gamma^\nu]\psi &: \text{ Anti-symmetric tensor}\\
\bar{\psi}\gamma_5\psi &: \text{ Pseudoscalar}\\
\bar{\psi}\gamma_5\gamma^\mu\psi &: \text{ Pseudovector}
\end{aligned}$$

1. *Dirac Conjugation for Matrices*

Similarly we can define Dirac conjugation for matrices

$$\bar{X} = \gamma^0X^\dagger\gamma^0$$

for some matrix X. Some useful identities are

$$\bar{\gamma}^\mu = \gamma^\mu$$

and

$$(\bar{u}_1Xu_2)^* = \bar{u}_2\bar{X}u_1$$

2. *Slashed Quantities*

To shorten our notation we define slashed quantities

$$\begin{aligned}
\gamma^\mu\partial_\mu &\to \not{\partial}\\
\gamma^\mu p_\mu &\to \not{p}\\
(\gamma^\mu p_\mu)(\gamma^\nu p_\nu) &\to (\not{p})^2 = p^2\\
(i\gamma^\mu\partial_\mu - m) &\to (i\not{\partial} - m)
\end{aligned}$$

C. Spin & Lorentz Transformations

Let us introduce the relativistic spin matrices

$$\Sigma_j = \frac{i}{4}\varepsilon_{jk\ell}\gamma^k\gamma^\ell = \frac{i}{8}\varepsilon_{jk\ell}[\gamma^k, \gamma^\ell]$$

We showed that

$$\mathbf{\Sigma}^2 = \frac{1}{2}\left(1 + \frac{1}{2}\right), \qquad [\Sigma_j, \Sigma_k] = i\varepsilon_{jk\ell}\Sigma_\ell$$

which implies that the Dirac Equation describes spin-$1/2$ particles. The spin alone does not commute with the Dirac Hamiltonian, it is the total angular momentum $\boldsymbol{J} = \boldsymbol{L} + \mathbf{\Sigma}$ that is a conserved quantity.

However, the Dirac spinor has four components as it describes a positive energy state, together with a negative energy state; an electron and a positron. However, even though we can write the spinor as a vector with four components:

$$\psi(x) = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \\ \psi_3(x) \\ \psi_4(x) \end{pmatrix}$$

it is *not* the case that spinors transform under Lorentz transformations in the same was a four- vectors do. Regular four-vector transform as follows:

$$x^\mu \to \Lambda^\mu_\nu x^\nu$$

however, spinors transform as spinors:

$$\psi \to \exp\left(\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}\right)\psi$$

where $\omega_{\mu\nu}$ contains all information about the Lorentz-angles, and

$$S^{\mu\nu} \equiv \frac{i}{4}[\gamma^\mu, \gamma^\nu]$$

However, we need the Dirac Equation to be Lorentz invariant, which means we need to establish how the the rest of the Dirac Equation changes under Lorentz transforms, such that the entire equation is invariant. In the lectures we found this by simplifying this by only look at the Weyl Equation.

D. Weyl Equation

The Weyl-Equation is the zero-mass limit of the Dirac Equation, where the Dirac Equation becomes block diagonal and it splits into two independent equations

$$i\sigma^\mu\partial_\mu\psi_R = 0, \qquad i\bar{\sigma}^\mu\partial_\mu\psi_L = 0$$

where

$$\sigma^\mu \equiv (\mathbb{1}, \boldsymbol{\sigma}), \qquad \bar{\sigma}^\mu \equiv (\mathbb{1}, -\boldsymbol{\sigma})$$

It turns out that there exists a group of matrices that describe the transformation of these Pauli vectors under Lorentz transformations

$$\sigma^\mu \to \Lambda^\mu_\nu\sigma^\nu = S\sigma^\mu S^\dagger$$

where $\det S = 1$. It turns out that $\Psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$ transforms like

$$\Psi \to \begin{pmatrix} S_L & 0 \\ 0 & S_R \end{pmatrix}\Psi$$

where $S_L(\beta, \phi) = S(\beta, \phi)$ and $S_R(\beta, \phi) = S(-\beta, \phi)$ and

$$S(\mathrm{d}\boldsymbol{\beta}, \mathrm{d}\boldsymbol{\phi}) = \mathbb{1} + a\mathrm{d}\boldsymbol{\beta}\cdot\boldsymbol{\sigma} + ib\mathrm{d}\boldsymbol{\phi}\cdot\boldsymbol{\sigma}, \qquad a, b \in \mathbb{R}$$

for some infinitesimal rotation parametrised by $\mathrm{d}\boldsymbol{\beta}$ and $\mathrm{d}\boldsymbol{\phi}$. β parametrises the boost part and ϕ the rotational part of the transformation.

E. Parity Transformation

Under parity transformations we go from a left spinor to a right spinor. This also implies that S_R and S_L transform into each other under parity transformations. Equation II D also holds in the massive case; The left and right spinors aren't mixed by Lorentz transformations.

F. Lorentz Algebra

The Lorentz algebra describes the generators of the Lorentz group, that is

$$\Lambda^\alpha{}_\beta \approx \delta^\alpha{}_\beta + \frac{i}{2}\omega_{\mu\nu}(M^{\mu\nu})^\alpha{}_\beta$$

where the generators $M^{\mu\nu}$ satisfy

$$[M^{\mu\nu}, M^{\rho\sigma}] = i\left(\eta^{\nu\rho}M^{\mu\sigma} - \eta^{\mu\rho}M^{\nu\sigma} + \eta^{\mu\sigma}M^{\nu\rho} - \eta^{\nu\sigma}M^{\mu\rho}\right)$$

the generators are antisymmetric in their indices $M^{\mu\nu} = -M^{\nu\mu}$. There are 6 generators, but they split into two set of three generators; those with only spatial indices generate rotations (L_k), whereas those with a temporal index generate boosts (K_k).

$$M^{\mu\nu} = \begin{pmatrix} 0 & K_1 & K_2 & K_3 \\ -K_1 & 0 & L_3 & -L_2 \\ -K_2 & -L_3 & 0 & L_1 \\ -K_3 & L_2 & -L_1 & 0 \end{pmatrix}$$

$$L_k \equiv \frac{1}{2}\varepsilon_{ijk}M^{ij}, \qquad K_i \equiv M^{i0}$$

The algebra has different representations, that can characterised by a pair of (half)-integers, (j, j'). The lowest of which are

$$\underbrace{(0,0)}_{\text{scalar}}, \quad \underbrace{\left(\frac{1}{2},0\right), \quad \left(0,\frac{1}{2}\right)}_{\text{spinor}}, \quad \underbrace{\left(\frac{1}{2},\frac{1}{2}\right)}_{\text{vector+scalar}}$$

For the spinor representation we denote the generators as $S^{\mu\nu}$, these are the same $S^{\mu\nu}$ as the ones defined previously.

G. Electromagnetic Interactions

Similarly to non-relativistic quantum mechanics the substitution from kinematic to canonical momentum $p_\mu \to p_\mu - eA_\mu$ accounts for electromagnetic interactions (at least if the right Gauge is chosen), hence

$$(p_\mu\gamma^\mu - eA_\mu\gamma^\mu - m)\psi = 0$$

Written as a Hamiltonian

$$H_{\text{Dirac,A}} = H_{\text{Dirac}} + eA_0 - e\boldsymbol{\alpha}\cdot\boldsymbol{A}$$

H. Solving the Free Dirac Equation, Rest-Frame

In the rest-frame* the Dirac Equation just becomes

$$\begin{pmatrix} 0 & m \\ m & 0 \end{pmatrix}\begin{pmatrix} u_L \\ u_R \end{pmatrix} = E\begin{pmatrix} u_L \\ u_R \end{pmatrix}$$

There are two solutions

$$\begin{aligned} E &= m, & u_L &= u_R \\ E &= -m, & u_L &= -u_R \end{aligned}$$

where the spinor $u_L = \xi$ is arbitrary. There are two linearly independent and orthogonal spinors ξ, the simplest example is

$$\xi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \xi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

So the positive energy solutions are

$$\psi_s^+(\boldsymbol{x},t) = e^{-imt}\begin{pmatrix} \xi_s \\ \xi_s \end{pmatrix}, \qquad s = 1,2$$

and the negative energy solutions are

$$\psi_s^-(\boldsymbol{x},t) = e^{imt}\begin{pmatrix} \eta_s \\ -\eta_s \end{pmatrix}, \qquad s = 1,2$$

generally ξ_s and η_s are two sets of linearly independent spinors. Using the z-component of the spin-operator defined in Equation II C we see that

$$\Sigma_z\psi_s^\pm = \begin{cases} +\frac{1}{2}\psi_s^\pm, & s = 1 \\ -\frac{1}{2}\psi_s^\pm, & s = 2 \end{cases}$$

hence for a general ξ_s the state describes a linear combination of up and down spin particles (anti-particles).

I. Spinors; Positive and Negative Modes

A general plane wave solution to the free Dirac equation is:

$$(i\partial\!\!\!/ - m)\psi(p)e^{-ip\cdot x} = e^{-ip\cdot x}(p\!\!\!/ - m)\psi(p) = 0$$

which only has solutions if the particle is on-shell, *i.e* if

$$\det(p\!\!\!/ - m) = (p^2 - m^2)^2 = (E^2 - \boldsymbol{p}^2 - m^2)^2$$

so

$$E = \pm\sqrt{\boldsymbol{p}^2 + m^2}$$

hence we can decompose the planar wave solution into the positive and negative modes

$$\psi(x) = u(\boldsymbol{p})e^{-ip\cdot x} + v(\boldsymbol{p})e^{+ip\cdot x}, \quad p_\mu = (\sqrt{\boldsymbol{p}^2 + m^2}, \boldsymbol{p})$$

applying the Dirac Hamiltonian

$$\begin{aligned} &(i\partial\!\!\!/ - m)\psi(x) \\ &= e^{-ip\cdot x}(p\!\!\!/ - m)u(\boldsymbol{p}) - e^{+ip\cdot x}(p\!\!\!/ + m)v(\boldsymbol{p}) = 0 \end{aligned}$$

due to the fact that the two exponentials are linearly independent we know that their coefficients must both be equal to zero, hence

$$\begin{aligned} (p\!\!\!/ - m)u(\boldsymbol{p}) &= 0 \\ (p\!\!\!/ + m)v(\boldsymbol{p}) &= 0 \end{aligned}$$

which also holds independently for each spin, s.

$$\boxed{\begin{aligned} (p\!\!\!/ - m)u_s &= 0, & \bar{u}_s(p\!\!\!/ - m) &= 0 \\ (p\!\!\!/ + m)v_s &= 0, & \bar{v}_s(p\!\!\!/ + m) &= 0 \end{aligned}}$$

* For a specific representation of the Clifford Algebra

1. Orthonormality

The spinors u and v must be orthogonal and normalised:

$$\bar{u}_s u_{s'} = 2m\delta_{ss'}, \quad \bar{v}_s v_{s'} = -2m\delta_{ss'}$$

A similar identity holds when we are dealing with massless particles

$$u_s^\dagger(p) u_{s'}(p) = 2E(p)\delta_{ss'}, \quad v_s^\dagger(p) v_{s'}(p) = 2E(p)\delta_{ss'}$$

If you're missing a minus in Equation III1, that is because v has *negative* energy. Hence if we want to require that $\psi_s(x)$ still describes a probability amplitude, we must have that, for example

$$\psi_s(x) = \frac{1}{\sqrt{2EV}} u_s e^{-ip\cdot x}$$

where we've assumed that the spinor part u_s satisfies Equation III1.

J. Charge Conjugation

Observe that $v^* e^{-ip\cdot x}$ has *positive* energy, however, it is not the case that ψ^* satisfies the positive-energy Dirac equation. Instead we need to transform

$$\psi \to -i\gamma^2\psi^* \equiv \psi^c$$

to go from a negative energy state to a positive energy state (and vice-versa). As the name hints towards, the charge conjugation transformation flips the sign of the charge of a particle, *i.e* if

$$\left(i\partial - e\not{A} - m\right)\psi = 0 \implies \left(i\partial + e\not{A} - m\right)\psi^c = 0$$

III. PERTURBATION THEORY

A. Fermi's Golden Rule & Coulomb Scattering

A *central* quantity in Relativistic Perturbation Theory is the number of transitions per unit time, which can be calculated by means of Fermi's Golden Rule:

$$\mathrm{d}w_{f\leftarrow i} = 2\pi \left|\langle f|\mathcal{V}|i\rangle\right|^2 \delta(E_f^{(0)} - E^{(0)} - \Omega)\mathrm{d}\Pi_f$$

where $\mathcal{V}$ is some perturbation and Ω is the driving frequency: we've assumed that $\mathcal{V}$ is a harmonic perturbation with frequency Ω. We have also introduced the density of final states

$$\mathrm{d}\Pi_f = g\frac{V\mathrm{d}^3\boldsymbol{k}_f}{(2\pi)^3}$$

where V is some volume and g is the internal degeneracy ($g = 2$ for spin-1/2 particles). Fermi's Golden Rule is often quoted in the continuous limit, where we take $\Omega \to 0$

$$\boxed{\mathrm{d}w_{f\leftarrow i} = 2\pi \left|\mathcal{V}_{fi}\right|^2 \delta(E_f^{(0)} - E^{(0)})\mathrm{d}\Pi_f}$$

In the case of the Dirac Equation, an electromagnetic perturbation is given by

$$\mathcal{V} = e\gamma^0\gamma^\mu A_\mu(x)$$

the simplest example is if we consider a state electric field;

$$\mathcal{V} = eA_0(x)\not{\mathbb{K}}, \qquad A_0(x) = \frac{Ze}{4\pi\,|x|}$$

Our initial and final states are

$$\psi_i(x) = \sqrt{\frac{1}{2E_iV}} u_s(p_i)e^{-ip_i\cdot x},$$
$$\bar{\psi}_f(x) = \sqrt{\frac{1}{2E_fV}} \bar{u}_r(p_f)e^{ip_f\cdot x}$$

where $p_i = (E_i, \boldsymbol{p}_i)$, $p_f = (E_f, \boldsymbol{p}_f)$. This gives us the following expression for the matrix element

$$\mathcal{V}_{fi} = \frac{Ze}{4\pi}\frac{1}{V}\sqrt{\frac{1}{4E_iE_f}}\bar{u}_f(p_f)\gamma^0 u_i(p_i)\frac{4\pi}{|\boldsymbol{q}|^2}$$

where $\boldsymbol{q} = \boldsymbol{p}_f - \boldsymbol{p}_i$ is the transferred momentum, and the last term came from the Fourier transform of the Coulomb potential. In reality we are interested in the squared matrix element

$$|\mathcal{V}_{fi}|^2 = \frac{1}{4E_iE_fV^2}\left|\bar{u}_f(p_f)\gamma^0 u_i(p_i)\right|^2 \left|\frac{4\pi Z\alpha}{\boldsymbol{q}^2}\right|^2$$

where $\alpha = \frac{e^2}{\hbar c}$ is the fine-structure constant. We see that this interaction is of order α^2.

1. Spin Sums & The Controversial Trace-Trick

In Equation III A the initial spinor has spin s whereas the final has spin r, which gives us a matrix-element of the form

$$|\mathcal{M}|^2 = \left|\bar{u}_r\gamma^0 u_s\right|^2$$

it is standard to average over initial sums and sum over final sums. We denote that we have performed this spin sum by setting a bar over the matrix element, *i.e*

$$\overline{|\mathcal{M}|^2} \equiv \frac{1}{2}\sum_s \sum_r \left|\bar{u}_r \gamma^0 u_s\right|^2$$

Additionally, $\bar{u}_r \gamma^0 u_s \in \mathbb{C}$, which means that

$$\mathrm{Tr}\left[\left|\bar{u}_r \gamma^0 u_s\right|^2\right] = \left|\bar{u}_r \gamma^0 u_s\right|^2$$

Using the cyclicity of the trace we can instead write

$$\left|\bar{u}_r \gamma^0 u_s\right|^2 = \mathrm{Tr}\left[u_r \bar{u}_r \gamma^0 u_s \bar{u}_s \gamma^0\right]$$

Note that $u_s \bar{u}_s$ is a matrix, which once we sum over spins becomes

$$\sum_s u_s(p)\bar{u}_s(p) = \not{p} + m, \qquad \sum_s v_s(p)\bar{v}_s(p) = \not{p} - m$$

Thus the spinor part of Equation III A becomes

$$\overline{|u_r(p_f)\gamma^0 u_s(p_i)|^2} = \frac{1}{2}\,\mathrm{Tr}\left[\gamma^0(m + \not{p}_i)\gamma^0(m + \not{p}_f)\right]$$

now we can use the γ-matrix trace identities from Equations 1 to obtain

$$\overline{|u_r \gamma^0 u_s|^2} = 4E_i^2\left(1 - \frac{|\boldsymbol{p}_i|^2}{E_i^2}\sin^2\left(\frac{\theta}{2}\right)\right)$$

where we've used energy conservation to set the two energies equal. To obtain this result we used Equation II A.

B. Cross-Section

The cross-section is defined as the number of particles scattered into solid angle $\mathrm{d}\Omega$ divided by the total flux of incoming particles. The flux of a single particle is

$$\boldsymbol{\Phi} = \frac{\bar{u}_s \boldsymbol{\gamma} u_s}{2E_i V} = \frac{1}{V}\frac{\boldsymbol{p}_i}{E_i} = \frac{\boldsymbol{v}}{V}$$

The cross-section is then

$$\mathrm{d}\sigma = \frac{\mathrm{d}w_{f\leftarrow i}}{|\boldsymbol{\Phi}|}$$

again looking at the Coulomb scattering example we get that

$$\mathrm{d}w_{f\leftarrow i} = \frac{1}{V}\frac{4Z^2\alpha^2}{\boldsymbol{q}^4}\left(1 - \frac{\boldsymbol{p}_i^2}{E_i^2}\sin^2\left(\frac{\theta}{2}\right)\right)|\boldsymbol{p}_i|\,E_i \mathrm{d}\Omega$$

and hence

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{4Z^2\alpha^2}{\boldsymbol{q}^4}\left(1 - \frac{\boldsymbol{p}_i^2}{E_i^2}\sin^2\left(\frac{\theta}{2}\right)\right)$$

We can attempt to write this in a more general form, by recognising that a part of the expression came from the Fourier transform of the potential

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{E^2}{4\pi^2}\left|eA_0(\boldsymbol{q})\right|^2\left(1 - \beta^2\sin^2\left(\frac{\theta}{2}\right)\right), \qquad \beta \equiv \frac{|\boldsymbol{p}_i|}{E_i}$$

in the non-relativistic limit, $\beta \to 0$, this reduced to the classical Rutherford formula.

C. Scattering off an Arbitrary Potential

Consider now a general electromagnetic scattering process, where the matrix element becomes

$$\mathcal{V}_{fi} = \frac{1}{V}\sqrt{\frac{1}{4E_i E_f}}\,\bar{u}_r(p_f)\,\gamma^\mu\, u_s(p_i)\int \mathrm{d}^3\boldsymbol{x}\; A_\mu\,(x)e^{i\boldsymbol{x}\cdot(\boldsymbol{p}_i - \boldsymbol{p}_f)}$$

thus we are now interested in

$$\begin{aligned}\frac{1}{2}\sum_{rs}\left(\bar{u}_r\gamma^\mu u_s\right)\left(\bar{u}_r\gamma^\nu u_s\right)^* &= \frac{1}{2}\,\mathrm{Tr}\left[\gamma^\mu(\not{p}_i + m)\gamma^\nu(\not{p}_f + m)\right] \\ &= 2\left[\eta^{\mu\nu}m^2 + p_i^\mu p_f^\nu + p_i^\nu p_f^\mu - \eta^{\mu\nu}(p_i\cdot p_f)\right]\end{aligned}$$

this gives us $\mathcal{V}_{fi}$ and hence we can calculate $\mathrm{d}w_{f\leftarrow i}$.

D. Electron-Muon Scattering

Unfortunately, in order to describe electron-muon scattering we need to make our methods slightly more sophisticated, this is because the matrix element

$$\langle\text{electron}|\text{interaction}|\text{muon}\rangle = 0$$

always, irrespective of the interaction. That is, electrons and muons do not interact directly, their interaction must be mediated by a particle that they both interact with; the photon (for example). Our states are now multi-particle states, we must keep track of the number of electrons, muons and photons:

$$|\Psi\rangle \equiv \left|e^-\right\rangle \otimes |\gamma\rangle \otimes \left|\mu^-\right\rangle$$

the initial and final states are

$$\begin{aligned}|i\rangle &= |e(p)\rangle \otimes |0\rangle \otimes |\mu(q)\rangle \\ |f\rangle &= |e(p')\rangle \otimes |0\rangle \otimes |\mu(q')\rangle\end{aligned}$$

however, due to the fact that the particles do not interact directly our interaction must involve an intermediate state

$$|n\rangle = |e(p_n)\rangle \otimes |\gamma(k_n)\rangle \otimes |\mu(q_n)\rangle$$

Due to the intermediate state(s) we need to use the second order equivalent of Fermi's Golden Rule:

$$\mathrm{d}w_{f\leftarrow i} = 2\pi \left| \sum_n \frac{\langle f|V|n\rangle \, \langle n|V|i\rangle}{E_i - E_n} \right|^2 \delta(E_i - E_f)\mathrm{d}\Pi_f$$

the sum is over all possible intermediate states. The interacting Dirac Hamiltonian in this case is

$$H = \boldsymbol{\alpha}\cdot\boldsymbol{p} + \beta m - e\beta\gamma^\mu A_\mu(x)$$

where

$$A_\mu(x) = \sum_\lambda \int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3\sqrt{\omega_{\boldsymbol{k}}}} \left[\varepsilon_\mu^{(\lambda)} a_{\boldsymbol{k}} e^{-ik\cdot x} + \varepsilon_\mu^{(\lambda)*} a_{\boldsymbol{k}}^\dagger e^{+ik\cdot x}\right]$$

where the two polarisation vectors satisfy $\varepsilon_\mu^{(\lambda)} \cdot k^\mu = 0$ and $\varepsilon_\mu^{(\lambda)} \cdot \left(\varepsilon^\mu_{(\lambda')}\right)^* = -\delta_{\lambda\lambda'}$.

However, this holds both for the electron and the muon, so A_μ is the quantity that couples the otherwise independent particles. We will denote the γ-matrices for the electron as γ^μ and for the muon as Γ^μ. One should think of these as

$$\gamma^\mu = \gamma^\mu \otimes \not{1}, \qquad \Gamma^\mu = \not{1} \otimes \gamma^\mu$$

i.e that they are the same quantities but live in different Hilbert Spaces; however these Hilbert Spaces are mixed due to the interaction. Define the matrix element $\mathcal{M}_1$ as

$$\mathcal{M}_1 \equiv \sum_\lambda \int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3} \frac{V_{in}V_{nf}}{E_i - E_n}$$

which can be shown to be equal to

$$\mathcal{M}_1 = \frac{(2\pi)^3\delta^{(3)}(\boldsymbol{p}+\boldsymbol{q}-\boldsymbol{p}'-\boldsymbol{q}')}{8\omega_{\boldsymbol{k}}V^2\sqrt{E_eE_e'E_\mu E_\mu'}} e^{i(E_i-E_f)t} \times \sum_\lambda \varepsilon_\mu\varepsilon_\nu^* \frac{(\bar{u}_s\gamma^\mu u_{s'})\left(\bar{U}_r\Gamma^\nu U_{r'}\right)}{(E_e - E_e') - \omega_{\boldsymbol{k}}}\Bigg|_{\boldsymbol{k}=\boldsymbol{q}'-\boldsymbol{q}=\boldsymbol{p}-\boldsymbol{p}'}$$

we can ignore the $e^{i(E_i-E_f)t}$ due to the energy conservation delta function in Fermi's Golden Rule. We need to square this quantity and then use all of our tricks to simplify this. Another useful identity is that

$$\sum_{\text{polarisations}} \varepsilon_\mu\varepsilon_\nu^* \to -\eta^{\mu\nu}$$

this is not an equality, but we can replace the polarisation sum by the Minkowski metric and still obtain the same answers.

The term $\mathcal{M}_1$ corresponds to the emission of a photon by the electron (and hence absorption by the muon), however the opposite process is also possible, so we define

$$\mathcal{M}_2 = \frac{(2\pi)^3\delta^{(3)}(\boldsymbol{p}+\boldsymbol{q}-\boldsymbol{p}'-\boldsymbol{q}')}{8\omega_{\boldsymbol{k}}V^2\sqrt{E_eE_e'E_\mu E_\mu'}} e^{i(E_i-E_f)t} \times \sum_\lambda \varepsilon_\mu\varepsilon_\nu^* \frac{(\bar{u}_s\gamma^\mu u_{s'})\left(\bar{U}_r\Gamma^\nu U_{r'}\right)}{-(E_e - E_e') - \omega_{\boldsymbol{k}}}\Bigg|_{\boldsymbol{k}=-(\boldsymbol{q}'-\boldsymbol{q})=-(\boldsymbol{p}-\boldsymbol{p}')}$$

Due to the fact that $\omega_{\boldsymbol{k}} = |\boldsymbol{k}|$ the photon energies are (naturally) the same in the two processes, thus the total matrix element is

$$\mathcal{M} = \mathcal{M}_1 + \mathcal{M}_2 = (2\pi)^3\delta^{(3)}(\boldsymbol{p}+\boldsymbol{q}-\boldsymbol{p}-\boldsymbol{q}')\times \frac{(\bar{u}_s\gamma^\mu u_{s'})}{2V\sqrt{E_eE_e'}} \frac{\left(\bar{U}_r\Gamma_\mu U_{r'}\right)}{2V\sqrt{E_\mu E_\mu'}} \frac{e^2}{(p-p')^2}$$

notice that the last product in the expression above is the Green's function of the wave equation. Now we need to square the matrix element and perform the spin sum†. We obtain

$$\overline{|\mathcal{M}|^2} = \frac{(2\pi)^6}{16V^4} \frac{I^{(e)}_{\mu\nu}}{E_eE_e'E_\mu E_\mu'} \frac{I^{\mu\nu}_{(\mu)}}{(p-p')^4} \left[\delta^{(3)}(\boldsymbol{p}+\boldsymbol{q}-\boldsymbol{p}'-\boldsymbol{q}')\right]^2$$

where we have defined

$$I^{(e)}_{\mu\nu} \equiv \frac{1}{2}\,\mathrm{Tr}\left[\gamma_\mu(m+\not{p})\gamma_\nu(m+\not{p}')\right]$$
$$I^{\mu\nu}_{(\mu)} \equiv \frac{1}{2}\,\mathrm{Tr}\left[\Gamma^\mu(M+\not{q})\Gamma^\nu(M+\not{q}')\right]$$

the squared delta-function gives us a factor V once we integrate. Putting this altogether we get hat

$$\mathrm{d}w_{f\leftarrow i} = \frac{(2\pi)^4}{V} \frac{e^4 I^{(e)}_{\mu\nu} I^{\mu\nu}_{(\mu)}}{2E_e\,2E_\mu} \frac{\delta^{(4)}(p+q-p'-q')}{(p-p')^4} \mathrm{d}\Pi_{\text{LIPS}}$$

where

$$\mathrm{d}\Pi_{\text{LIPS}} = \frac{\mathrm{d}^3\boldsymbol{p}'}{(2\pi)^3(2E_e')} \frac{\mathrm{d}^3\boldsymbol{q}'}{(2\pi)^3(2E_\mu')}$$

Equation III D is almost Lorentz invariant, it is only the volume and the $2E_e\,2E_\mu$ that is non-Lorentz invariant. This issue is solved by looking at the cross-section instead; the flux in this case us

$$\Phi = \frac{1}{V}\left|\frac{\boldsymbol{p}}{E_e} - \frac{\boldsymbol{q}}{E_\mu}\right|$$

† here we have two initial spins and two final spins so the spin sum corresponds to $\frac{1}{4}\sum_{srs'r'}$

which can be written as

$$VE_1E_2\Phi = \sqrt{(p\cdot q)^2 - m^2M^2}$$

In the limit where the electron is relativistic we can simplify the I terms:

$$I^2 = 8M^2\left[(E_e - E_e')^2 + 2E_eE_e'\cos^2\frac{\theta}{2}\right]$$

putting this all together and integrating with respect to the muon momentum and energy we obtain

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{\alpha^2}{4E_e^2\sin^4\theta/2}\frac{E_e'}{E_e}\left(\cos^2\frac{\theta}{2} + \frac{Q^2}{2M^2}\sin^2\frac{\theta}{2}\right)$$

where

$$Q^2 = 2ME_e\left(1 - \frac{M}{M + 2E_e\sin^2\theta/2}\right)$$

E. Feynman Diagrams

Feynman Diagrams are tools that we will use to simplify the process of obtaining an expression for the squared matrix element. The convenient thing is that complicated processes can be expressed pictorially, and that the complicated processes can then quite easily be converted to a mathematical expression.

For a $2 \to N \geq 2$ process we have that

$$\mathrm{d}\sigma = \frac{(2\pi)^4\delta^{(4)}\left(p_1 + p_2 - \sum_{i=1}^N p_i'\right)}{4\sqrt{(p_1\cdot p_2)^2 - m_1^2m_2^2}}\boxed{|\mathcal{M}|^2}\prod_{i=1}^N\frac{\mathrm{d}^3\boldsymbol{p}_i'}{(2E_i')(2\pi)^3}$$

all of these quantities are assumed to be known expect the squared matrix element; Feynman Diagrams help us find this. Similarly if we want to calculate a decay rate for a $1 \to N \geq 2$ process we have

$$\mathrm{d}\Gamma = \frac{(2\pi)^4}{2M}\delta^{(4)}\left(P - \sum_{i=1}^N p_i'\right)\boxed{|\mathcal{M}|^2}\prod_{i=1}^N\frac{\mathrm{d}^3\boldsymbol{p}_i'}{(2E_i')(2\pi)^3}$$

for an initial particle with mass M and $P = (M, 0, 0, 0)$ (at rest)

1. Feynman Rules for Quantum Electrodynamics

We need to distinguish between internal and external lines. External lines are lines that are only connected to the diagram at one of its two vertices. Internal lines are connected at both vertices and describe virtual particles.

1. a. Each incoming spinor is denoted by $u_s, U_r, \dots$
   b. Each incoming photon is denoted by $\varepsilon_\mu, \dots$
2. a. Each outgoing spinor is denoted by $\bar{u}_{s'}, \bar{U}_{r'}, \dots$
   b. Each outgoing photon is denoted by $\varepsilon_\mu^*, \dots$
3. At each vertex write $(-ie\gamma^\mu)$ for $\mu, \nu, \dots$
4. Start at the *end* of a fermion line and follow the lines backwards. Write whatever appears from left to right
5. Internal lines get *propagators:*

$$\text{photons}: \quad \frac{-i\eta_{\mu\nu}}{q^2 + i\varepsilon}$$

$$\text{fermions}: \quad \frac{i(\not{q} + m)}{q^2 - m^2 + i\varepsilon}$$

   where q is chosen such that momentum is conserved at each vertex and the photon indices match the γ^μ indices at the vertices from before
6. Set this all equal to $i\mathcal{M}$

Tabel I: *Feynman Rules: QED*

IV. FERMI THEORY

Prior to Fermi theory β-decay wasn't properly understood; one assumed that β-decay occurred due to a contact force between for instance an incoming neutron and the three products, a proton an electron and an anti-electron-neutrino.

Though this model had some promising predictions (that fit very well to data) there was a fundamental flaw with the model: the cross section did not conform to the unitarity (Froissart) bound which implies there was something fundamentally wrong with the way of thinking about what was later to become weak interactions. This will lead us to the prediction and discovery of the W-boson.

Before we describe this, let us talk about parity violation. All theory we have learnt thus far is invariant under parity transformations ($\boldsymbol{x} \to -\boldsymbol{x}$, $\boldsymbol{E} \to -\boldsymbol{E}$, $\boldsymbol{B} \to \boldsymbol{B}$, $\boldsymbol{p} \to -\boldsymbol{p}$, $\boldsymbol{S} \to -\boldsymbol{S}$ etc.). However, parity is violated by weak interactions, instead it is charge-parity that is a symmetry of weak interactions[‡]. This tells us that left & right particles are distinct for weak

[‡] It turns out that $\mathcal{CP}$-symmetry also is violated

interactions; in fact the weak force only couples to left particles.

By introducing a massive particle, the W-boson we can solve the unitarity issue, which adds the following to the Feynman Rules

1. *c.* Each incoming W-boson is denoted by $\varepsilon_\mu, \ldots$

2. *c.* Each outgoing W-boson is denoted by $\varepsilon^*_\mu, \ldots$

3. At each weak-interaction-vertex write

$$\frac{ig}{2\sqrt{2}}\gamma^\mu(\mathbb{1} - \gamma_5)$$

4. Same as before

5. Internal lines get *propagators:*

$$W\text{-bosons}: \quad -i\frac{\eta_{\mu\nu} - \frac{p_\mu p_\nu}{M_W^2}}{q^2 - M_W^2 + i\varepsilon}$$

$$\text{neutrinos}: \quad i\frac{\not{q}}{q^2}$$

where q is chosen such that momentum is conserved at each vertex and the photon indices match the γ^μ indices at the vertices from before

6. Same as before

Tabel II: *Feynman Rules: W-bosons*

W-boson spin sums give us

$$\sum_\lambda \varepsilon_\mu \varepsilon^*_\nu = \left(-\eta_{\mu\nu} + \frac{k_\mu k_\nu}{M_W^2}\right)$$

The coupling constant g is called the *weak coupling constant* and is related to the Fermi-g factor through

$$\frac{8G_F}{\sqrt{2}} = \frac{g^2}{M_W^2 c^4}, \qquad G_F \approx 1.16 \times 10^{-5}\mathrm{GeV}^{-2}$$

A. Mandelstam Variables

Figur 1: *A generic $2 \to 2$ process*

Consider a $2 \to 2$ process. It is useful to define

$$s = (p_1 + p_2)^2$$
$$t = (p_1 - k_1)^2$$
$$u = (p_1 - k_2)^2$$

For instance if

$$=$$

we get[§] that the momentum of the boson in the middle has momentum $k^2 = s$, which is why this is called the s-channel diagram. Similarly if

$$=$$

the boson has momentum $k^2 = t$ – this is the t-channel[¶]. Finally

$$=$$

we have $k^2 = u$ – the u-channel. This is just like the t-channel except that the last two particles were swapped around.

B. Gauge Invariance and Gauge Fields

As we have seen in regular quantum mechanics, we would like our observables to be invariant under Gauge transformations of the form

$$\psi(x) \rightsquigarrow \psi(x)e^{ie\Gamma(x)}$$

which implies the presence of an extra gauge field which transforms as

$$\boldsymbol{A} \rightsquigarrow \boldsymbol{A} + \nabla\Gamma(x)$$

which means that we should replace partial derivatives to the *covariant derivative*, similarly to in General Relativity

$$\partial_\mu \rightsquigarrow \mathrm{D}_\mu \equiv \partial_\mu - ieA_\mu$$

which implies that the Dirac Equation transforms as

$$\left[i\gamma^\mu\partial_\mu - m\right]\psi = 0 \rightsquigarrow \left[i\gamma^\mu(\partial_\mu - ieA_\mu) - m\right]\psi = 0$$

[§] In the relativistic limit $\overline{|\mathcal{M}|^2} = 8e^4\frac{t^2+u^2}{s^2}$

[¶] In the relativistic limit $\overline{|\mathcal{M}|^2} = 8e^4\frac{s^2+u^2}{t^2}$

This implies that simply because we require our physics to be invariant under gauge transformations we see that our particles interact with a new field, this is known as *minimal coupling* or *gauge interactions*.

It is natural to define the slashed covariant derivative, just like we did for other quantities

$$\not{D} \equiv \gamma^\mu (\partial_\mu - ieA_\mu(x))$$

in which case the Gauge invariant Dirac Equation becomes

$$i\not{D}\psi - m\psi = 0$$

the covariant derivative does not in general commute with itself,

$$[D_\mu, D_\nu] = -ieF_{\mu\nu}$$

which allows us to write the Klein-Gordon Equation for a spinor in a covariant form

$$\left(D_\mu D^\mu + m^2 - ieS^{\mu\nu}F_{\mu\nu}\right)\psi = 0$$

we can in fact do something similar for vector-bosons, such as the W-bosons. For them we have that

$$\left(W_\mu^+\right)^* = W_\mu^-$$

these vector fields transform as

$$W_\mu^\pm \rightsquigarrow e^{\pm ie\Gamma(x)} W_\mu^\pm$$

It turns out that the Klein-Gordon Equation for these vector bosons is

$$(D_\nu D^\nu + M_W^2)W_\mu^\pm - ie\kappa F_{\mu\nu}W_\pm^\nu$$

for some unknown dimensionless κ. We see that W-bosons also interact solely with photons (not only with fermions).

C. Electroweak Feynman Rules

1. The interaction $WW\gamma$ connects three vector fields $W_\mu^+(p), W_\lambda^-(k), A_\nu(q)$ thus the vertex should carry those three indices
2. We choose all momenta to be outgoing, thus
$$p_\mu + q_\mu + k_\mu = 0$$
3. The resulting vertex has the form
$$V_{\lambda\mu\nu}(k,p,q) = ie\left[(k-p)_\nu\eta_{\mu\lambda} + (p_\lambda - \kappa q_\lambda)\eta_{\mu\nu} + (\kappa q_\mu - k_\mu)\eta_{\nu\lambda}\right]$$

Tabel III: *Feynman Rules:* $W^+W^-\gamma$

Similarly for $WW\gamma\gamma$.

$$V_{\mu\nu\rho\sigma} = -e^2\left(2\eta_{\mu\nu}\eta_{\rho\sigma} - \eta_{\mu\rho}\eta_{\nu\sigma} - \eta_{\mu\sigma}\eta_{\nu\rho}\right)$$

Tabel IV: *Feynman Rules:* $W^+W^-\gamma\gamma$

An example of the application of these rules is the process $e^+e^- \to W^+W^-$:

Figur 2: *One of the Feynman Diagrams for the* $e^+e^- \to W^+W^-$ *process*

The amplitude for which is given by

$$i\mathcal{M} = \frac{-g^2}{8}\bar{v}_s(p)\gamma^\mu(\not{1} - \gamma_5)\frac{i\not{q}}{q^2}\gamma^\nu(\not{1} - \gamma_5)u_r(k)\varepsilon_\mu^*(r_1)\varepsilon_\nu^*(r_2)$$

V. YANG-MILLS THEORY

Consider two distinct fermions, $\psi^{(1)}$ and $\psi^{(2)}$, with the same mass (or similar masses?). They live in each their Hilbert space and both satisfy a Dirac Equation:

$$(H_{\text{Dirac}} \oplus H_{\text{Dirac}})\,\boldsymbol{\psi} = 0, \qquad \boldsymbol{\psi} = \begin{pmatrix}\psi^{(1)} \\ \psi^{(2)}\end{pmatrix}$$

$\boldsymbol{\psi}$ is referred to as an iso-vector. We definitely need to require the probability to be conserved:

$$P = \int \mathrm{d}^3\boldsymbol{x}\;\boldsymbol{\psi}^\dagger \cdot \boldsymbol{\psi}$$

which works if we transform the iso-vector by means of a special-unitary (SU(2)) transformation. However, the question is whether this transformation maps solutions to the Dirac Equation to solutions. This is no different from what we did with Gauge Transformations, however the Gauge Transformation is a U(1) transformation and this is a SU(2) transformation. So is it the case that

$$(H_{\text{Dirac}} \oplus H_{\text{Dirac}})\, U(x)\boldsymbol{\psi} = 0, \qquad U(x) \in \text{SU}(2)$$

Let us denote $H_{\text{Dirac}} = (i\gamma^\mu \partial_\mu - m)$ (omitting the $\mathbb{1}_{2\times 2}\otimes$). If we let $U(x) = \exp(i\boldsymbol{\alpha}(x)\cdot \boldsymbol{t})$ where $\boldsymbol{t} = \boldsymbol{\sigma}/2$ then we get that

$$\left[i\partial\!\!\!/ - m\right] U(x)\boldsymbol{\psi} = U(x)\left[i\partial\!\!\!/ - \partial\!\!\!/\alpha(x) - m\right]\boldsymbol{\psi} = 0$$

which is reminiscent of what we obtained for Gauge transformations. We defined

$$\partial\!\!\!/\alpha(x) = \gamma^\mu \partial_\mu \alpha(x) = \gamma^\mu \partial_\mu \boldsymbol{\alpha}(x)\cdot \boldsymbol{t} = -i\gamma^\mu U^\dagger \partial_\mu U$$

As we did previously we should now introduce a covariant derivative that accounts for this extra term. Define now

$$gB_\mu^{(j)} = -\partial_\mu \alpha^{(j)}$$

Then the modified Dirac Equation reads

$$\left(i\gamma^\mu\partial_\mu + g\gamma^\mu \underbrace{\boldsymbol{B}_\mu \cdot \boldsymbol{t}}_{\equiv \mathbb{B}_\mu} - m\right)\boldsymbol{\psi} = 0$$

defining $\mathbb{B}_\mu$ makes our notation a bit more compact. Thus the covariant derivative should be

$$\mathbb{D}_\mu \equiv \partial_\mu \mathbb{1}_{2\times 2} - ig\mathbb{B}_\mu$$

which means

$$\left[i\mathbb{D}\!\!\!/_\mu - m\right]\boldsymbol{\psi} = 0$$

is our new Dirac Equation. An important identity is that

$$\mathbb{D}'_\mu \boldsymbol{\psi}' = \mathbb{D}'_\mu U(x)\boldsymbol{\psi} = U(x)\mathbb{D}_\mu \boldsymbol{\psi}$$

where the transformed $\mathbb{B}_\mu$ is

$$\mathbb{B}'_\mu = U\mathbb{B}_\mu U^\dagger - \frac{i}{g}\left(\partial_\mu U\right)U^\dagger$$

A. Non-abelian Field Strength

For electrodynamics we had Equation IV B

$$F_{\mu\nu} \equiv \frac{i}{e}[D_\mu, D_\nu]$$

and similarly for our non-abelian fields we have

$$\mathbb{G}_{\mu\nu} = \frac{i}{g}[\mathbb{D}_\mu, \mathbb{D}_\nu] = \partial_\mu \mathbb{B}_\nu - \partial_\nu \mathbb{B}_\mu - ig[\mathbb{B}_\mu, \mathbb{B}_\nu]$$

The covariant derivative acting on the field gives us

$$\mathbb{D}_\lambda \mathbb{G}_{\mu\nu} = \partial_\lambda \mathbb{G}_{\mu\nu} - ig[\mathbb{B}_\lambda, \mathbb{G}_{\mu\nu}]$$

thus $\mathbb{G}_{\mu\nu}$ is not Gauge invariant which points towards the fact that these non-abelian fields are *charged.* If we were to define the Yang-Mills Lagrangian now we would get

$$\mathcal{L}_{\text{Yang-Mills}} = -\frac{1}{2}\operatorname{Tr}\left(\mathbb{G}_{\mu\nu}\mathbb{G}^{\mu\nu}\right)$$

which is the Lagrangian of a massless field, however, we observe only massive fields. We need the Higgs Mechanism to understand how these particles obtain Mass.

CONTENTS