I. SPECIAL RELATIVITY

Spacetime consists of events whose coordinates x^μ are four-vectors:

$$x^\mu = (ct, x^1, x^2, x^3)$$

however we set $c = 1$ which means that time is measured in units of length. The Minkowski metric is the metric of free space, which can in a suitable coordinate system be written as

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Using this metric we define the proper length:

$$\mathrm{d}s^2 = \eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu$$

which is invariant under

- *Translations*

$$x^\mu \rightsquigarrow x^\mu + a^\mu$$

- *Rotations*

$$x^\mu \rightsquigarrow \Lambda^\mu_\nu x^\nu, \qquad \Lambda^\mu_\nu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

  i.e. rotations about spatial axes.

- *Boosts*

$$x^\mu \rightsquigarrow \Lambda^\mu_\nu x^\nu, \qquad \Lambda^\mu_\nu = \begin{pmatrix} \cosh\phi & -\sinh\phi & 0 & 0 \\ -\sinh\phi & \cosh\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

  where the boost parameter, ϕ, is related to the velocity difference between the initial and final inertial systems:

$$\tanh\phi = v$$

  which allows us to write this in a more familiar way:

$$\Lambda^\mu_\nu = \begin{pmatrix} \gamma & -\gamma v & 0 & 0 \\ -\gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \gamma \equiv \left(1 - v^2\right)^{-\frac{1}{2}}$$

We differentiate between line segments depending on the sign of $\mathrm{d}s^2$:

- *Spacelike separation:* $\mathrm{d}s^2 > 0$
- *Null separation:* $\mathrm{d}s^2 = 0$
- *Timelike separation:* $\mathrm{d}s^2 < 0$

For timelike curves we can define the proper time:

$$\mathrm{d}\tau^2 = -\mathrm{d}s^2 = -\eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu$$

Which allows us to define the relativistic velocity:

$$u^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} = \gamma(1, v^1, v^2, v^3)$$

Note that

$$u_\mu u^\mu = \frac{1}{1 - v^2}\left(v^2 - 1\right) = -1$$

we can also define the relativistic acceleration:

$$a^\mu = \frac{\mathrm{d}u^\mu}{\mathrm{d}\tau} = \frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\tau^2}$$

Curves

We have up till now dealt with infinitesimal line segments; we will now generalise proper lengths and proper times to curves. Given a curve $x^\mu(\lambda)$:

$$\Delta\tau = \int_{\lambda_1}^{\lambda_2} \mathrm{d}\lambda \sqrt{-\eta_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}}$$

II. EQUIVALENCE PRINCIPLE

> EINSTEIN'S EQUIVALENCE PRINCIPLE:
> In small enough regions of spacetime the laws of physics reduce to those of special relativity. Therefore, it is impossible to measure the presence of gravity through local experiments.

Due to this equivalence principle we choose to measure acceleration with respect to *freely falling particles.*

A. Gravitational Redshift

Before we introduce the language of general relativity, we can already show a physical effect due to the principle of equivalence, which contradicts the predictions made by Newtonian physics:

Consider a tower of height h, where a laser at the bottom of the tower emits light upwards and the wavelength of the light is measured at the top. Due to the fact that both the top and the bottom of the tower are accelerating upwards the wavelength at the bottom of the tower will be different from that at the top, due to the finite speed of light:

$$\Delta\lambda = \lambda_{\text{top}} - \lambda_{\text{bottom}} = \frac{a_g h}{c^2}\lambda_{\text{bottom}}$$

Generally we can express this as

$$\frac{\Delta\lambda}{\lambda} = \frac{1}{c^2}\Delta\phi$$

where $\Delta\phi$ is the potential difference between the two points.

This also has the implication that gravitational fields dilate time:

$$cT_{\text{bottom}} = \lambda_{\text{bottom}}, \qquad cT_{\text{top}} = \lambda_{\text{top}}$$

hence

$$\Delta T = \frac{\Delta\phi}{c^2}T_{\text{bottom}}, \qquad \frac{T_{\text{top}}}{T_{\text{bottom}}} = 1 + \frac{\Delta\phi}{c^2}$$

which tells us that clocks go slower inside gravitational wells.

III. GENERAL SPACETIMES

The Minkowski metric is only valid when there is no gravity (except for local inertial systems, that is). As we have seen, gravity influences the passage of time, and it also influences relative lengths. Therefore we need a more general metric than the Minkowski metric: $\eta_{\mu\nu} \rightsquigarrow g_{\mu\nu}$. Hence

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu$$

$g_{\mu\nu}$ is a tensor, therefore it transforms as a tensor:

> Transformation of the metric
> Suppose we are given a (bijective) coordinate transform:
>
> $$\tilde{x}^\mu = \tilde{x}^\mu(x^\nu)$$
>
> Then the metric, which is a $(0,2)$-tensor, transforms like
>
> $$\tilde{g}_{\alpha\beta} = \frac{\mathrm{d}x^\mu}{\mathrm{d}\tilde{x}^\alpha}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tilde{x}^\beta}g_{\mu\nu}$$

The inverse metric $g^{\mu\nu}$ is defined as the tensor that satisfies

$$g_{\mu\nu}g^{\nu\rho} = \delta^\rho_\mu$$

which transforms like a $(2,0)$-tensor:

> Transformation of the inverse-metric
> Suppose we are given a (bijective) coordinate transform:
>
> $$\tilde{x}^\mu = \tilde{x}^\mu(x^\nu)$$
>
> Then the inverse metric, which is a $(2,0)$-tensor, transforms like
>
> $$\tilde{g}^{\alpha\beta} = \frac{\partial\tilde{x}^\alpha}{\partial x^\mu}\frac{\partial\tilde{x}^\beta}{\partial x^\nu}g^{\mu\nu}$$

Geodesics

Freely falling particles follow curves that maximise proper time; these paths are called geodesics. Using this principle we can, using the calculus of variations find a differential equation that describes geodesics.

We want to find the curve, $x^\mu(\lambda)$ that minimises:

$$\Delta\tau\left[x^\mu(\lambda)\right] = \int_{\lambda_1}^{\lambda_2}\mathrm{d}\lambda\,\sqrt{-g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}}$$

hence we require $\delta\Delta\tau = 0$, which results in the following equation

> Geodesic Equation
> The motion of a freely falling particle is described by a curve, $x^\mu(\tau)$, which satisfies:
>
> $$\frac{\mathrm{d}^2x^\mu}{\mathrm{d}\tau^2} + \Gamma^\mu_{\alpha\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau}\frac{\mathrm{d}x^\beta}{\mathrm{d}\tau} = 0$$
>
> where the Christoffel Symbol is defined by
>
> $$\Gamma^\mu_{\alpha\beta} = \frac{1}{2}g^{\mu\sigma}\left(\partial_\alpha g_{\beta\sigma} + \partial_\beta g_{\alpha\sigma} - \partial_\sigma g_{\alpha\beta}\right)$$

The Christoffel Symbol is symmetric in its lower indices.

NB: The Christoffel symbol is *not* a tensor!

Newtonian Limit

When we look at very weak, time-independent gravity ($g_{\mu\nu} \approx \eta_{\mu\nu} + h_{\mu\nu}(x)$ for $h_{\mu\nu}$ small), and we require low velocities $\left(\left|\frac{\mathrm{d}x^i}{\mathrm{d}\tau}\right| \ll 1\right)$, general relativity should simplify to Newton's law of gravitation. In this limit the geodesic equation is

$$\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\tau^2} + \Gamma^\mu_{00}\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 = 0$$

Up to linear order in $h_{\mu\nu}$ this becomes

$$\frac{\mathrm{d}^2 \boldsymbol{x}}{\mathrm{d}\tau^2} = \frac{1}{2}\nabla h_{00}$$

hence we can interpret the time-time entry in the metric as the gravitational potential in the Newtonian limit:

$$g_{00} \approx -1 - 2\phi, \qquad \text{Newtonian Limit}$$

A. Local Inertial System

LOCAL INERTIAL SYSTEM
Consider an event, p. There always exists a coordinate system in which the metric simplifies to the Minkowski metric as per Equation III. This coordinate system is referred to as a *Local Inertial System*, for which we have

$$g_{\mu\nu}\big|_p = \eta_{\mu\nu}, \qquad \partial_\rho\, g_{\mu\nu}\big|_p = 0$$

This implies that in the local inertial system $\Gamma^\rho_{\mu\nu} = 0$, which ties back to the implication of Einstein's Equivalence Principle, that we cannot detect the presence of gravity *locally.*

IV. TENSORS AND THE PRINCIPLE OF COVARIANCE

The simplest tensor is a tensor with zero indices, a scalar. Scalars transform rather trivially, consider $\tilde{x}^\mu = \tilde{x}^\mu(x^\mu)$:

$$\tilde{\Phi}(\tilde{x}^\mu) = \Phi(x^\mu)$$

The next simplest tensor has one index, which we can either place high or low. If it is placed high the vector is a contravariant vector, V^μ, which we also just call a vector. If we place the index down low it is a covariant vector, or a dual vector, V_μ. These vectors are related to each other:

$$V_\mu = g_{\mu\nu} V^\nu$$

These quantities transform quite differently from scalar fields:

$$\tilde{V}_\mu = \frac{\partial x^\nu}{\partial \tilde{x}^\mu} V_\nu, \qquad \tilde{V}^\mu = \frac{\partial \tilde{x}^\mu}{\partial x^\nu} V^\nu$$

Generally though

TENSOR TRANSFORMATION
Given a $(m,n)-$tensor $T^{\mu_1\cdots\mu_m}{}_{\nu_1\cdots\nu_n}$ it transforms as

$$\tilde{T}^{\alpha_1\cdots\alpha_m}{}_{\beta_1\cdots\beta_n} = \prod_{(i,j)=(1,1)}^{(m,n)} \frac{\partial \tilde{x}^{\alpha_i}}{\partial x^{\mu_i}} \frac{\partial x^{\nu_j}}{\partial \tilde{x}^{\beta_j}} T^{\mu_1\cdots\mu_m}{}_{\nu_1\cdots\nu_n}$$

A. Covariant Derivative

We need to generalise the concept of a derivative, so that we can express our equations in a covariant form; in a form that is independent of coordinate system. This is very easy for scalar fields, because their derivatives are already covariant:

$$\tilde{\partial}_\mu \tilde{\Phi} = \frac{\partial x^\nu}{\partial \tilde{x}^\mu} \partial_\nu \Phi$$

however, this is not the case generally. For instance, let us look at how the derivative of a vector field, $\partial_\mu V^\nu$, transforms:

$$\begin{aligned}\tilde{\partial}_\alpha \tilde{V}^\beta &= \frac{\partial x^\mu}{\partial \tilde{x}^\alpha}\partial_\mu\left(\frac{\partial \tilde{x}^\beta}{\partial x^\nu} V^\nu\right)\\ &= \frac{\partial x^\mu}{\partial \tilde{x}^\alpha}\frac{\partial \tilde{x}^\beta}{\partial x^\nu}\partial_\mu V^\nu + \frac{\partial x^\mu}{\partial \tilde{x}^\alpha}\frac{\partial^2 \tilde{x}^\beta}{\partial x^\mu \partial x^\nu} V^\nu\\ &= \frac{\partial x^\mu}{\partial \tilde{x}^\alpha}\frac{\partial \tilde{x}^\beta}{\partial x^\nu}\left(\partial_\mu V^\nu + \frac{\partial x^\nu}{\partial \tilde{x}^\gamma}\frac{\partial^2 \tilde{x}^\gamma}{\partial x^\mu \partial x^\rho} V^\rho\right)\end{aligned}$$

The second term is the problem, if this term were equal to zero we would have shown that the partial derivative of a vector field is covariant. Let us define the covariant derivative, D_μ, which is the generalisation of the partial derivative, which reduces to the partial derivative in a local inertial system. In a local inertial system it can be shown that

$$\Gamma^\nu_{\mu\rho}\big|_{\mathrm{LIS,p}} = \frac{\partial x^\nu}{\partial \tilde{x}^\gamma}\frac{\partial^2 \tilde{x}^\gamma}{\partial x^\mu \partial x^\rho}$$

We are now ready to define the covariant derivative of vector field:

COVARIANT DERIVATIVE OF A VECTOR-FIELD
Given a general spacetime, $g_{\mu\nu}$, and a vector field, V^ρ, the covariant derivative is defined by

$$\mathrm{D}_\mu V^\rho = \partial_\mu V^\rho + \Gamma^\rho_{\mu\sigma} V^\sigma$$

where $\Gamma^\rho_{\mu\sigma}$ is the Christoffel Symbol defined in Equation III

The covariant derivative of a dual-vector is almost the same

> **Covariant Derivative of a Dual-vector-field**
> Given a general spacetime, $g_{\mu\nu}$, and a vector field, V_ρ, the covariant derivative is defined by
>
> $$\mathrm{D}_\mu V_\rho = \partial_\mu V_\rho - \Gamma^\sigma_{\mu\rho} V_\sigma$$
>
> where $\Gamma^\rho_{\mu\sigma}$ is the Christoffel Symbol defined in Equation III

The covariant derivative of a general (m, n)-tensor is

$$\begin{aligned}\mathrm{D}_\rho T^{\mu_1\cdots\mu_m}{}_{\nu_1\cdots\nu_n} &= \partial_\rho T^{\mu_1\cdots\mu_m}{}_{\nu_1\cdots\nu_n} + \Gamma^{\mu_1}_{\rho\sigma} T^{\sigma\mu_2\cdots\mu_m}{}_{\nu_1\cdots\nu_n} \\ &+ \Gamma^{\mu_2}_{\rho\sigma} T^{\mu_1\sigma\mu_3\cdots\mu_m}{}_{\nu_1\cdots\nu_n} + \cdots + \Gamma^{\mu_m}_{\rho\sigma} T^{\mu_1\cdots\mu_{m-1}\sigma}{}_{\nu_1\cdots\nu_n} \\ &- \Gamma^\sigma_{\rho\nu_1} T^{\mu_1\cdots\mu_m}{}_{\sigma\nu_2\cdots\nu_n} - \Gamma^\sigma_{\rho\nu_2} T^{\mu_1\cdots\mu_m}{}_{\nu_1\sigma\nu_3\cdots\nu_n} \\ &- \cdots - \Gamma^\sigma_{\rho\nu_n} T^{\mu_1\cdots\mu_m}{}_{\nu_1\cdots\nu_{n-1}\sigma}\end{aligned}$$

1. *Covariant derivative along a curve*

Consider a curve $x^\mu(\lambda)$, then we can calculate the covariant derivative along this curve. For instance the covariant derivative of a vector along the curve would be

$$\frac{\mathrm{D}}{\mathrm{d}\lambda} V^\mu \equiv \frac{\mathrm{d}x^\rho}{\mathrm{d}\lambda} D_\rho V^\mu$$

If the covariant derivative of a quantity along a curve is zero, then the quantity is conserved along the curve. For example if V^μ is parallel to u^μ at the beginning of the curve and the quantity $V_\mu u^\mu$ is conserved along the curve, then we can conclude that the vectors remain parallel along the curve.

2. *Properties of the covariant derivative*

The covariant derivative, like other derivatives, has

- linearity
- a product rule

> **Covariant derivative of the metric**
> Given a general spacetime with metric $g_{\mu\nu}$ and inverse metric $g^{\mu\nu}$, then we have
>
> $$D_\rho g_{\mu\nu} = D_\rho g^{\mu\nu} = 0$$
>
> which can easily be shown by transforming into a local inertial system.

B. Acceleration in General Relativity

In the special relativity section we defined the four-velocity, $u^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}$. We now define the covariant acceleration:

$$a^\mu = \frac{\mathrm{D}}{\mathrm{d}\tau} u^\mu = \frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} D_\nu u^\mu$$

Writing this out, using the definition of the covariant derivative:

$$\begin{aligned}a^\mu &= \frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\left(\partial_\nu u^\mu + \Gamma^\mu_{\nu\rho} u^\rho\right) \\ &= \frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\frac{\partial u^\mu}{\partial x^\nu} + \Gamma^\mu_{\nu\rho}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} u^\rho \\ &= \frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\tau^2} + \Gamma^\mu_{\nu\rho}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\rho}{\mathrm{d}\tau}\end{aligned}$$

This simplifies to the geodesic equation in the case where $a^\mu = 0$, i.e when there are no non-gravitational forces. Additionally, in a local inertial system $a^\mu = \frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\tau^2}$, which is the acceleration we defined for special relativity, as we would expect.

For massive particles travelling on geodesics we have

$$a^\mu = 0, \qquad u_\mu u^\mu = -1$$

however, for null curves (for which $\mathrm{d}\tau = 0$) we instead have

$$a^\mu = 0, \qquad g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda} = 0$$

where λ is the *affine parameter* we use to parametrise the null curve.

NB: τ cannot be used to parametrise the null curve, because it is constant along the curve.

C. Maxwell's Equations in a General Spacetime

Maxwell's equations in vacuum can be expressed as

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \qquad D_\mu F^{\mu\nu} = -J^\nu$$

We can see that the electromagnetic field strength tensor, $F_{\mu\nu}$, is antisymmetric. These equations are covariant, which is easy to see for the second equation. The appearance of partial derivatives in the first equation makes it look like this may not be covariant. However, due to the symmetry of the Christoffel symbols in their lower indices we have

$$D_\mu A_\nu - D_\nu A_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu$$

Assuming a particle is only subject to the electromagnetic force then its covariant acceleration can be calculated through

$$a^\mu = -\frac{q}{m} g_{\nu\rho} u^\nu F^{\rho\mu}$$

V. RIEMANN CURVATURE TENSOR

We are looking for an object that describes the curvature of the metric, $g_{\mu\nu}$, and hence of spacetime in a manner that all coordinate systems agree upon: we are looking for a covariant expression of the curvature. Unfortunately our best candidate, $\Gamma^\rho_{\mu\nu}$, cannot be used, because it is not a tensor*

We instead will look at a derivative of the Christoffel symbol†, but we wish for a covariant expression.

Consider an infinitesimal square with (infinitesimal) width w^μ and (infinitesimal) height h^μ, with starting position x_0^μ. We wish to transport a vector V^μ along the curve in such a way that the covariant derivative is and remains zero, this process is referred to as parallel transport. We, therefore, demand that

$$\mathrm{d}V^\mu = -\Gamma^\mu_{\rho\nu} V^\nu \mathrm{d}x^\rho$$

We will ignore terms of higher order than h^2 (and hence w^2 and hw)

Let us write this in a more legible way

$$V^\mu(x_0 + w) - V^\mu(x_0) = -\Gamma^\mu_{\rho\nu}(x_0) V^\nu(x_0) w^\mu$$

going to the next corner

$$\begin{aligned}\Delta V_1 = V^\mu(x_0 + w + h) - V^\mu(x_0) = -\Gamma^\mu_{\rho\nu}(x_0) V^\nu(x_0) w^\rho \\ -\Gamma^\mu_{\rho\nu}(x_0 + w) V^\nu(x_0 + w) h^\rho\end{aligned}$$

---

* It does not transform like a tensor. This is easy to show, $\Gamma^\rho_{\mu\nu} = 0$ in a local inertial system, but nonzero generally which cannot hold for a tensor, see Equation IV.

† We cannot just use $D_\sigma \Gamma^\rho_{\mu\nu}$ because this is not a tensor either

But we can write this out

$$\begin{aligned}\Delta V_1 = &-\Gamma^\mu_{\rho\nu}(x_0) V^\nu(x_0) w^\rho \\ &- \left(\Gamma^\mu_{\rho\nu}(x_0) + \partial_\alpha \Gamma^\mu_{\rho\nu}(x_0) w^\alpha\right) \left(V^\nu(x_0) - \Gamma^\nu_{\alpha\beta} V^\alpha(x_0) w^\beta\right) h^\rho\end{aligned}$$

we can also get there the other way, so first adding h and then w:

$$\begin{aligned}\Delta V_2 = &-\Gamma^\mu_{\rho\nu} V^\nu h^\rho \\ &- \left(\Gamma^\mu_{\rho\nu} + \partial_\alpha \Gamma^\mu_{\rho\nu} h^\alpha\right) \left(V^\nu - \Gamma^\nu_{\alpha\beta} V^\alpha h^\beta\right) w^\rho\end{aligned}$$

The difference $\Delta V = \Delta V_1 - \Delta V_2$ is also the difference between the two parallel transported vectors (along the different paths). By renaming some of the dummy variables we end up with

$$\Delta V^\mu = \underbrace{\left(\partial_\rho \Gamma^\mu_{\sigma\nu} - \partial_\sigma \Gamma^\mu_{\rho\nu} - \Gamma^\mu_{\sigma\alpha}\Gamma^\alpha_{\nu\rho} + \Gamma^\mu_{\rho\alpha}\Gamma^\alpha_{\nu\sigma}\right)}_{R^\mu{}_{\nu\rho\sigma}} V^\nu w^\rho h^\sigma$$

RIEMANN CURVATURE TENSOR
The Riemann Curvature Tensor is defined as

$$R^\rho{}_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\alpha}\Gamma^\alpha_{\sigma\nu} - \Gamma^\rho_{\nu\alpha}\Gamma^\alpha_{\sigma\mu}$$

Clearly $R^\rho{}_{\sigma\mu\nu} = 0$ if and only if spacetime is locally flat.

A. Symmetries of the Riemann Curvature Tensor

$R^\rho_{\sigma\mu\nu}$ is antisymmetric in μ and ν as swapping these indices would correspond to switching the two paths discussed in the derivation. To discuss further symmetries let us lower the upper index:

$$R_{\mu\nu\rho\sigma} = g_{\mu\alpha} R^\alpha{}_{\nu\rho\sigma}$$

we have that

$$R_{\mu\nu\rho\sigma} = -R_{\nu\mu\rho\sigma} = -R_{\mu\nu\sigma\rho}$$

and

$$R_{\mu\nu\rho\sigma} = R_{\rho\sigma\mu\nu}$$

and finally

$$R_{\mu\nu\rho\sigma} + R_{\mu\sigma\nu\rho} + R_{\mu\rho\sigma\nu} = 0$$

Ricci Tensor and Scalar

We can contract the first and third indices of the Riemann tensor, to obtain the Ricci tensor:

$$R_{\mu\nu} = g^{\alpha\beta} R_{\alpha\mu\beta\nu} = R^\alpha{}_{\mu\alpha\nu}$$

which is a symmetric tensor

The trace of the Ricci tensor is the Ricci scalar:

$$g^{\mu\nu}R_{\mu\nu} = R = R^\alpha_\alpha$$

B. Bianchi Identity

The following identity holds for the Riemann curvature tensor:

$$D_\alpha R_{\mu\nu\rho\sigma} + D_\nu R_{\alpha\mu\rho\sigma} + D_\mu R_{\nu\alpha\rho\sigma} = 0$$

Due to the fact that the covariant derivative of the metric is zero we can freely contract indices inside the Bianchi Identity:

$$\begin{aligned} 0 &= g^{\nu\sigma}g^{\mu\rho}\left(D_\alpha R_{\mu\nu\rho\sigma} + D_\nu R_{\alpha\mu\rho\sigma} + D_\mu R_{\nu\alpha\rho\sigma}\right) \\ &= D_\alpha R - 2D^\sigma R_{\sigma\alpha} \end{aligned}$$

Therefore, for the Ricci tensor we can compress (some of?) the information from the Bianchi Identity into a simpler equation:

$$D^\mu\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right) = 0$$

VI. EINSTEIN'S EQUATIONS

We've already seen that the metric is the general relativistic counterpart of the Newtonian gravitational potential, therefore we might be inclined to assume that the Riemann curvature tensor is the general relativistic counterpart of the Laplacian of the potential. This is essentially the case, but as we will see it suffices to use the Ricci tensor.

However, we now need to find the general relativistic counterpart of the mass density. This object is clearly a $(0,2)$-tensor, like the metric, and in order for the Newtonian limit to make any sense we can already state that T_{00} will simplify to the mass density in the Newtonian limit. This object is defined below.

> Energy-Momentum Tensor
> The energy-momentum tensor, $T^{\mu\nu}(x)\prod_{\rho=0,\rho\neq\nu}\mathrm{d}x^\rho$, is the flux of the μ^{th} component of the four-momentum across an infinitesimal 3-surface $\prod_{\rho=0,\rho\neq\nu}\mathrm{d}x^\rho$.

This definition is difficult to interpret, so let us write this more concretely

T^{00} : Energy density
T^{i0} : Density of the i^{th} component of the momentum
T^{0j} : Energy flux through the surface orthogonal to x^j
T^{ij} : Internal forces per unit area (pressure and shear)

Energy flux is the same as momentum density ($c = 1$), and similarly for T^{ij} and T^{ji}. Therefore, the energy-momentum tensor is symmetric

$$T^{\mu\nu} = T^{\nu\mu}$$

In the Newtonian limit we already know what happens[‡]:

$$T^{\mu\nu} = \begin{pmatrix} \rho_m & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

we call this kind of matter *Newtonian matter*. A perfect fluid (zero viscosity and zero heat conduction) has the following energy-momentum tensor:

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + pg_{\mu\nu}$$

The specific type of fluid can be expressed through the function $p = p(\rho)$:

Dust: $p = 0$
Photons: $p = \frac{1}{3}\rho$

A. Conservation of the Energy-Momentum Tensor

In special relativity we have the identity

$$\partial_\mu T^{\mu\nu} = 0$$

this is a continuity equation: the (temporal) change in energy is due to energy flux out of a volume element.

We can quite easily generalise this to a covariant equation:

$$D_\mu T^{\mu\nu} = 0$$

indeed, the energy-momentum tensor is conserved. The metric is also covariantly conserved, so we can raise and lower indices, which implies

$$D^\mu T_{\mu\nu} = 0$$

[‡] This is because everyday pressures (like 1atm) are $\sim 10^{-12}c^2\rho_m$, hence we can readily neglect these

B. Einstein's Equations

We now have two conserved quantities, $T_{\mu\nu}$ and $R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$, thus these quantities must be equal up to an "integration"[§] constant:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}$$

where Λ is a free parameter, which later leads to the concept of dark energy and the expansion of the universe. The Λ-term can be thought of as a zero-point energy, i.e. even when $T_{\mu\nu} = 0$ there is energy in the universe, which is why it can make sense to write the term on the right hand side. We can show that the $8\pi G$ term must be there if we want to use our prior definition of $T_{\mu\nu}$; it appears when we take the Newtonian limit.

VII. SCHWARZSCHILD METRIC

Let us consider a spherically symmetric mass distribution. This symmetry must also apply to the metric, therefore we are interested in a line-segment that is spherically symmetric. Beginning with the most general line-segment in spherical coordinates:

$$\mathrm{d}s^2 = -A(t,r)\mathrm{d}t^2 + B(t,r)\mathrm{d}r^2 + 2C(t,r)\mathrm{d}t\mathrm{d}r + D(t,r)r^2\mathrm{d}\Omega^2$$

the functions A, B, C and D can only depend on time and r due to the aforementioned spherical symmetry. We can diagonalise the (t,r)-part by changing basis, and by redefining r we can also remove D. We additionally write g_{tt} and g_{rr} using exponentials, to simplify our calculations later:

$$\mathrm{d}s^2 = -e^{2\alpha(r)}\mathrm{d}t^2 + e^{2\beta(r)}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2$$

We have also removed the time dependence; even with time-dependence we will obtain the same result, as stated by *Birkhoff's theorem.* By plugging this metric into Einstein's equation and assuming that $T_{\mu\nu} = 0$ outside the spherically symmetric object we get differential equations that describe the functions α and β:

$$R_{tt} = e^{2(\alpha-\beta)}\left(\partial_r^2\alpha + (\partial_r\alpha)^2 - \partial_r\alpha\partial_r\beta + \frac{2}{r}\partial_r\alpha\right) = 0$$

$$R_{rr} = -\partial_r^2\alpha + \partial_r\alpha\partial_r\beta - (\partial_r\alpha)^2 + \frac{2}{r}\partial_r\beta = 0$$

$$R_{\theta\theta} = \frac{1}{\sin^2\theta}R_{\phi\phi} = 1 + e^{-2\beta}\left(r(\partial_r\beta - \partial_r\alpha) - 1\right)$$

We can take linear combinations of the elements of the Ricci tensor, giving us

$$\partial_r(\alpha + \beta) = 0$$

we can absorb the integration constant into our definition of t, so we can conclude that $\alpha = -\beta$. The final differential equation we need to solve is thus

$$1 - e^{2\alpha}(1 + 2r\partial_r\alpha) = 0$$

let $2\alpha = \ln(f)$:

$$f + r\partial_r f = 1 = \partial_r(rf)$$

hence

$$f = 1 - \frac{r_0}{r}$$

in conclusion

$$g_{\mu\nu} = \begin{pmatrix} -(1-\frac{r_0}{r}) & 0 & 0 & 0 \\ 0 & \left(1-\frac{r_0}{r}\right)^{-1} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix}$$

By once again looking at the Newtonian limit we can conclude $r_0 = 2GM$, the *Schwarzschild radius.*

A. Geodesics

Define

$$\dot{x}^\mu \equiv \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}, \quad \ddot{x}^\mu \equiv \frac{\mathrm{d}\dot{x}^\mu}{\mathrm{d}\tau}$$

This makes the geodesic equation a bit more palatable:

$$\ddot{x}^\mu + \Gamma^\mu_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta = 0, \qquad g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = -1$$

for massive particles following time-like curves. Consider now the Christoffel Symbols:

$$\Gamma^t_{tr} = \frac{1}{2}\partial_r\ln\left(1 - \frac{r_0}{r}\right), \qquad \Gamma^\theta_{r\theta} = \Gamma^\phi_{r\phi} = \frac{1}{r}$$
$$\Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta, \qquad \Gamma^\phi_{\theta\phi} = \cot\theta$$

There are other nonzero components, however, we will only use the aforementioned in the following. Using these we find that

$$\ddot{t} + \partial_r\ln\left(1 - \frac{r_0}{r}\right)\dot{t}\dot{r} = 0$$
$$\ddot{\theta} + \frac{2}{r}\dot{r}\dot{\theta} - \sin\theta\cos\theta\dot{\phi}^2 = 0$$
$$\ddot{\phi} + \frac{2}{r}\dot{r}\dot{\phi} + 2\cot\theta\dot{\theta}\dot{\phi} = 0$$

[§] Strictly speaking it is not an integration constant, just a tensor whose covariant derivative is also zero.

we can write these slightly differently though

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left[\dot{t}\left(1-\frac{r_0}{r}\right)\right]=0$$
$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(r^2\dot{\theta}\right)=r^2\sin\theta\cos\theta\dot{\phi}^2$$
$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(r^2\sin^2\theta\dot{\phi}\right)=0$$

and for the r component we instead use the formula for a line segment and use the normalisation of $\dot{x}^\mu$:

$$-1=-\left(1-\frac{r_0}{r}\right)\dot{t}^2+\frac{\dot{r}^2}{1-\frac{r_0}{r}}+r^2\left(\dot{\theta}^2+\sin^2\theta\dot{\phi}^2\right)$$

Restricting ourselves to $\theta=\frac{\pi}{2}$, which we can always do by rotating coordinate systems¶. Under this choice of coordinates we get

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left[\dot{t}\left(1-\frac{r_0}{r}\right)\right]=0$$
$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(r^2\dot{\phi}\right)=0$$
$$-1=-\left(1-\frac{r_0}{r}\right)\dot{t}^2+\frac{\dot{r}^2}{1-\frac{r_0}{r}}+r^2\dot{\phi}^2$$

The first two equations describe conserved quantities, the first being the energy and the second the angular momentum.

$$E=\dot{t}\left(1-\frac{r_0}{r}\right)$$
$$L=r^2\dot{\phi}$$

using which we can simplify the third equation significantly

$$E^2=1-\frac{2GM}{r}+\dot{r}^2+\frac{L^2}{r^2}\left(1-\frac{2GM}{r}\right)$$

This differs from the classical counterpart only by the final term, which goes like r^{-3}. We can change coordinate systems to $X=\frac{GM}{r}$ and express this as a differential equation in ϕ:

$$\frac{\mathrm{d}^2X}{\mathrm{d}\phi^2}+X-3X^2=\left(\frac{GM}{L}\right)^2$$

here it is the $3X^2$ that is the relativistic correction to Newton's equation. Define $Y=\left(\frac{L}{GM}\right)^2X$ we get

$$\frac{\mathrm{d}^2Y}{\mathrm{d}\phi^2}+Y-3\left(\frac{GM}{L}\right)^2Y^2=1$$

¶ The angular momentum vector is still a conserved quantity, as we will see.

hence the Newtonian limit clearly corresponds to $X\ll\left(\frac{GM}{L}\right)^2$, which is achieved by looking at sufficiently large values of r.

$$\frac{1}{r}\ll\frac{GM}{L^2}$$

In the intermediate limit, so where we can treat the Newtonian solution as the solution the unperturbed problem, and let the Y^2 term perturb the solution, we begin with

$$r(\phi)=\frac{(1-e^2)a}{1+e\cos\phi}$$

where

$$E^2=1+\left(\frac{GM}{L}\right)^2\left(e^2-1\right)\qquad\frac{L^2}{GM}=\left(1-e^2\right)a$$

e is the eccentricity of the orbit, and a is its semimajor axis. Thus we look at

$$X_0(\phi)=\left(\frac{GM}{L}\right)^2(1+e\cos\phi)$$

and calculate the first order correction

$$\frac{\mathrm{d}^2X_1}{\mathrm{d}\phi^2}+X_1=3X_0^2$$

giving us

$$X_1(\phi)=\left(\frac{GM}{L}\right)^4\left(3\left(1+\frac{e^2}{2}+e\phi\sin\phi\right)-\frac{e^2}{2}\cos2\phi\right)$$

the solution is no longer periodic, which is what explains Mercury's precession** Using this we can calculate the precession of an elliptic orbit per period:

$$\Delta\phi=\frac{6\pi GM}{(1-e^2)ac^2}$$

B. Deflection of Light

The geodesic equation for null curves also predicts things that aren't found classically, as we will see:

$$\ddot{x}^\mu+\Gamma^\mu_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta=0,\qquad g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu=0$$

** Actually, the largest part of Mercury's precession is accounted for classically; this just accounts for a small part of it.

however now we cannot use τ as a parameter any more, so for null curves we need

$$\dot{x}^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}, \quad \ddot{x}^\mu = \frac{\mathrm{d}\dot{x}^\mu}{\mathrm{d}\lambda}$$

The fact that these are null curves actually doesn't change that much. We still have energy and angular momentum conservation:

$$E = \frac{\mathrm{d}t}{\mathrm{d}\lambda}\left(1 - \frac{r_0}{r}\right), \quad L = r^2\frac{\mathrm{d}\phi}{\mathrm{d}\lambda}$$

and the differential equation describing X becomes

$$\frac{\mathrm{d}^2X}{\mathrm{d}\phi^2} + X = 3X^2$$

so the only difference is that we no longer have the $\left(\frac{GM}{L}\right)^2$ term. Notice that the differential equation in the absence of X^2 describes straight lines, as we might expect. However the X^2 term curves the straight lines. The deflection angle for a light ray with impact parameter, b, due to GR is

$$\Delta\phi = \frac{4GM}{bc^2}$$

VIII. BLACK HOLES

A. Schwarzschild Metric

We know that energy is conserved, which tells us that

$$\mathrm{d}\tau = \frac{1 - \frac{r_0}{r}}{E}\mathrm{d}t$$

Thus at $r = r_0$ the proper time appears to stop with respect to t[††]. Let us require that for $r \rightsquigarrow \infty$ the proper time and coordinate time agree with each other, which implies that the infalling object is at rest at infinity: $E = 1$. Thus

$$\frac{\mathrm{d}r}{\mathrm{d}\tau} = -\sqrt{\frac{r_0}{r}} \rightsquigarrow r(\tau)^{\frac{3}{2}} - r(\tau_0)^{\frac{3}{2}} = \frac{3}{2}\sqrt{r_0}\,(\tau_0 - \tau)$$

Thus according to the infalling observer nothing special happens at r_0; the only special point is at $r = 0$, which is the physical singularity. However, according to the distant observer we get

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{\mathrm{d}r}{\mathrm{d}\tau}\frac{\mathrm{d}\tau}{\mathrm{d}t} = -\left(1 - \frac{r_0}{r}\right)\sqrt{\frac{r_0}{r}}$$

[††] The distant observer measures t, and when comparing to τ the distant observer claims that time stops at the Schwarzschild radius.

close to $r = r_0$ we can approximate this as

$$\frac{\mathrm{d}r}{\mathrm{d}t} \approx -\frac{r - r_0}{r_0}$$

which is an exponential decay

$$r(t) \approx r_0 + r_C \exp\left(-\frac{t}{r_0}\right)$$

hence according to the distant observer the event horizon is never reached.

1. *Closing of Light Cones*

In spacetime diagrams in special relativity we always know that massive objects' motions are limited by the light cones. The same holds in general relativity, however, here the light cones are more complicated. In the following we look at radial null curves, which will limit the motions of massive objects that also are moving radially. Radial null curves have

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \pm\left(1 - \frac{r_0}{r}\right)$$

according to the distant observer. At $r \approx r_0$ we see that the apparent speed of light is zero, which again corresponds to the proper time stopping according to the distant observer. This implies that the infalling and outgoing null curves both have zero coordinate radial velocity at the Schwarzschild radius. Therefore no massive object appears to cross the event horizon in finite Schwarzschild coordinate time, according to the distant observer.

2. *Eddington-Finkelstein Coordinates*

As we have seen, the fact that time appears to stop at the event horizon is coordinate dependent. This is because the metric has a singularity at $r = r_0$ (and one at $r = 0$). The singularity at $r = r_0$ is an artefact of the coordinate system, *not* of the physics; therefore, there exist coordinate systems where this singularity is not present. However, the singularity at $r = 0$ cannot be mended with coordinate transforms; it is a singularity in the spacetime.

Let us now define a coordinate system that mends the $r = r_0$ singularity, by redefining the radial component

$$r_\star = r + r_0 \ln\left|\frac{r}{r_0} - 1\right|$$

clearly

$$\mathrm{d}r_\star = \frac{\mathrm{d}r}{1-\frac{r_0}{r}}$$

In these coordinates radial null curves obey

$$\frac{\mathrm{d}r_\star}{\mathrm{d}t} = \pm 1 \quad \rightsquigarrow \quad r_\star(t) = \pm t + r_\star(0)$$

thus using this as a radial coordinate will fix the problem we just saw with light cones closing. However the function $r(r_\star)$ is not particularly pretty, therefore, it is beneficial to work a bit harder for our new coordinate system. Define

$$u = t - r^\star, \qquad v = t + r^\star$$

thus for u and v constant we have radial null curves. The coordinate transform $(t,r) \rightsquigarrow (v,r)$ gives us the following line segment:

$$\mathrm{d}s^2 = -\left(1-\frac{r_0}{r}\right)\mathrm{d}v^2 + 2\mathrm{d}v\mathrm{d}r + r^2\mathrm{d}\Omega^2$$

clearly we have removed the singularity at $r = r_0$.

3. Inside the Black Hole

In these coordinates outgoing radial null curves are described by

$$\frac{\mathrm{d}v}{\mathrm{d}r} = \frac{2}{\left(1-\frac{r_0}{r}\right)}$$

and infalling null curves have $\mathrm{d}v = 0$. We see that inside the black hole both infalling and outgoing null curves move toward smaller r, which implies that light as well as massive objects are forced to travel towards the singularity if they are inside the event horizon. Interestingly, because there is a well-defined direction of the passage of r (always decreasing) we can think of r as a time-coordinate, whereas t becomes a spatial coordinate. This is also seen from the Schwarzschild metric, which is valid inside the black hole, as g_{tt} and g_{rr} have opposite signs inside and outside the black hole.

The singularity at the origin is a physical singularity in spacetime, therefore, we cannot remove it by changing coordinates. We can see this by finding a covariant quantity that describes the curvature of the spacetime, which diverges at the origin. We use the Kretschmann scalar for this:

$$R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} = \frac{12r_0^2}{r^6}$$

which clearly diverges at the origin.

4. Kruskal-Szekeres Coordinates

Our coordinate systems so far have not had global time coordinates; we needed to use different time coordinates inside and outside of the black hole. This is exactly what Kruskal-Szekeres coordinates solve. They are defined using the Eddington-Finkelstein quantities, u and v:

$$\mathcal{T} = \frac{1}{2}\left(e^{\frac{v}{2r_0}} - e^{-\frac{u}{2r_0}}\right), \quad \mathcal{R} = \frac{1}{2}\left(e^{\frac{v}{2r_0}} + e^{-\frac{u}{2r_0}}\right)$$

Using which we can write out the line-segment

$$\mathrm{d}s^2 = \frac{4r_0^3}{r}e^{-\frac{r}{r_0}}\left(-\mathrm{d}\mathcal{T}^2 + \mathrm{d}\mathcal{R}^2\right) + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2\right)$$

where $r(\mathcal{T},\mathcal{R})$ is defined through

$$\mathcal{T}^2 - \mathcal{R}^2 = \frac{r_0 - r}{r_0}e^{\frac{r}{r_0}}$$

There are four interesting parts of spacetime:

$$\begin{aligned} r = r_0 \quad &\leftrightarrow \quad \mathcal{R} = \pm\mathcal{T} \\ r > r_0 \quad &\leftrightarrow \quad |\mathcal{R}| > |\mathcal{T}| \\ r < r_0 \quad &\leftrightarrow \quad |\mathcal{R}| < |\mathcal{T}| \\ r = 0 \quad &\leftrightarrow \quad \mathcal{T}^2 = 1 + \mathcal{R}^2 \end{aligned}$$

the final part is rather strange; the singularity is now given by two hyperbolae.

5. Gravitational Redshift

Once again we use

$$\frac{\mathrm{d}t}{\mathrm{d}\tau} = \frac{E}{1-\frac{r_0}{r}}$$

putting this into the line-segment for a radially inward falling massive object:

$$\frac{\mathrm{d}r}{\mathrm{d}\tau} = \sqrt{E^2 - 1 + \frac{r_0}{r}}$$

The change in u and v are given by

$$\mathrm{d}u = \left(\frac{E + \sqrt{E^2-1+\frac{r_0}{r}}}{1-\frac{r_0}{r}}\right)\mathrm{d}\tau, \quad \mathrm{d}v = \left(\frac{E - \sqrt{E^2-1+\frac{r_0}{r}}}{1-\frac{r_0}{r}}\right)\mathrm{d}\tau$$

which we obtain by simply using the above expressions for $\mathrm{d}t$ and $\mathrm{d}r$.

The distant observer, who measures t, has constant $r = r_\infty$. Thus in this coordinate system $\Delta u = \Delta\tau_\infty = \Delta v$. However, r is not constant for the inward falling observer; we assume that it is approximately constant for the duration $\Delta\tau$. For an outgoing null curve u is constant, therefore we can conclude that Δu must be the same for the observer at infinity and the inward falling observer, thus

$$\Delta\tau_\infty = \left(\frac{E + \sqrt{E^2 - 1 + \frac{r_0}{r}}}{1 - \frac{r_0}{r}}\right)\Delta\tau$$

Thus $\Delta\tau_\infty > \Delta\tau$; there is a gravitational redshift.

Similarly, for a radially inward falling null curve v is constant, which means that the infalling observer experiences a change in period of light sent from the distant observer:

$$\Delta\tau_\infty = \left(\frac{E - \sqrt{E^2 - 1 + \frac{r_0}{r}}}{1 - \frac{r_0}{r}}\right)\Delta\tau$$

For $E > 1$ the factor on the right-hand side is always smaller than $\frac{1}{2}$, therefore for $E > 1$ we always have a gravitational redshift. However, for $E < \frac{1}{2}$ there is both blueshift and redshift, at the Schwarzschild radius there is always a blueshift for $E < \frac{1}{2}$.

Two fixed observers

If both the distant observer and the observer near the black hole have fixed positions we just get

$$\mathrm{d}\tau^2 = \left(1 - \frac{r_0}{r}\right)\mathrm{d}\tau_\infty^2$$

Thus there is a redshift for light sent *to* the distant observer, and blueshift for light sent *from* the distant observer.

Observations of Black Holes

We differentiate between two groups of black holes; stellar mass black holes with masses of $3m_\odot$ to $100m_\odot$, and supermassive black holes with masses ranging from $10^6 m_\odot$ to $10^{10} m_\odot$.

Stellar black holes are formed from collapsing remnants of supernovae. How supermassive black holes form is a bit of a mystery, though astrophysicists presume that they form from stellar black holes that collide and fuse, and slowly fall towards the centre of galaxies.

Black holes are difficult to observe directly, because they absorb all light. Before gravitational wave methods were developed we could only measure them indirectly, for example by looking at the trajectories of nearby stars.

B. Kerr Black Hole

The Schwarzschild metric is spherically symmetric, which means that it cannot rotate. Kerr black holes are allowed to rotate, so we no longer require a spherically symmetric metric. However, we still require that the metric is independent of time and that the metric is a solution to Einstein's equations.

The line-element given by the Kerr metric is

$$\begin{aligned}\mathrm{d}s^2 = &-\left(1 - \frac{r_0 r}{\Sigma}\right)\mathrm{d}t^2 - \frac{2ar_0 r}{\Sigma}\sin^2\theta \mathrm{d}t\mathrm{d}\phi \\ &+ \frac{(r^2+a^2)^2 - \Delta a^2 \sin^2\theta}{\Sigma}\sin^2\theta \mathrm{d}\phi^2 + \frac{\Sigma}{\Delta}\mathrm{d}r^2 + \Sigma \mathrm{d}\theta^2\end{aligned}$$

where

$$\Sigma = r^2 + a^2\cos^2\theta, \quad \Delta = r^2 - r_0 r + a^2$$

The event horizon is where g_{rr} blows up, as was the case for the Schwarzschild black hole. For the Kerr black hole this happens at

$$r_\pm = \frac{r_0}{2}\left(1 \pm \sqrt{1 - \frac{4a^2}{r_0^2}}\right)$$

The radius that is reached first by physical objects is r_+, therefore we treat that as the event horizon. The geodesic equation for null curves gives us an expression for the effective radial speed of light:

$$\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 = -\frac{g_{tt} + 2g_{t\phi}\frac{\mathrm{d}\phi}{\mathrm{d}t} + g_{\phi\phi}\left(\frac{\mathrm{d}\phi}{\mathrm{d}t}\right)^2 + g_{\theta\theta}\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^2}{g_{rr}}$$

at the event horizon $g_{rr} \to \infty$, whereas the other metric-elements remain finite; this implies that light cannot escape the black hole, just like what we saw for the Schwarzschild black hole.

1. Ergoregion and Frame Dragging

The g_{tt} element of the metric is zero when $\Sigma = r_0 r$, which is satisfied for

$$r_{\text{ergo}} = \frac{r_0}{2}\left(1 \pm \sqrt{1 - \frac{4a^2}{r_0^2}\cos^2\theta}\right)$$

the curve described by the + solution is outside of the event horizon, therefore, it is accessible to our universe. Interestingly enough, inside the ergoregion we have

$$-2g_{t\phi}\frac{\mathrm{d}\phi}{\mathrm{d}t} = \left(\frac{\mathrm{d}\tau}{\mathrm{d}t}\right)^2 + g_{tt} + g_{\phi\phi}\left(\frac{\mathrm{d}\phi}{\mathrm{d}t}\right)^2 + g_{rr}\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 + g_{\theta\theta}\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^2$$

the right hand side is positive (not non-negative!) and $g_{t\phi} < 0$, therefore we have that $\dot{\phi} > 0$; there is a well-defined direction in which everything inside the ergoregion must move, just like we saw *inside* the event horizon of the Schwarzschild black hole. However, the ergoregion is accessible to us, *and* it can be left. This effect is called frame dragging; the rotation of the Kerr black hole is forcing spacetime to corotate.

To further understand this, let us look at the parabola above. Specifically let us look at null curves that travel only in the ϕ direction, hence $\mathrm{d}r = 0$ and $\mathrm{d}\theta = 0$, and because they are null curves $\mathrm{d}\tau = 0$, thus

$$-g_{\phi\phi}\Omega^2 - 2g_{t\phi}\Omega = g_{tt}$$

hence

$$\Omega_\pm = \frac{-g_{t\phi} \pm \sqrt{g_{t\phi}^2 - g_{\phi\phi}g_{tt}}}{g_{\phi\phi}} = \frac{-g_{t\phi} \pm \sin\theta\sqrt{\Delta}}{g_{\phi\phi}}$$

clearly massive objects must move at angular frequencies between the two extremes:

$$\Omega_- < \Omega < \Omega_+$$

However as we decrease the distance from the centre toward r_+ the two frequencies will converge towards the same value, which we call the angular frequency of the black hole:

$$\Omega_H = \Omega_\pm(r_+, \theta) = \frac{a}{r_+ r_0}$$

C. Asymptotically Flat Metrics

Both the Schwarzschild metric and the Kerr metric are asymptotically flat; at large distances from the black hole we can approximate the metric by the Minkowski metric. This property is very important and makes intuitive sense: there may be a SMBH at the centre of a galaxy, but we do not expect it to prevent us from treating the space between the Earth and the Moon as approximately flat.

D. Black Hole Uniqueness

The Schwarzschild and Kerr metric have the following properties:

- They are asymptotically flat
- There is an event horizon
- Outside the event horizon the metric is a solution to the vacuum Einstein equations
- The metric is stationary

Amazingly there can exist no other metrics that satisfy these four conditions. This means that the two black holes we have discussed are the only two that satisfy these conditions and if there exist any other black holes in the universe it must be because at least one of these conditions is not satisfied. Thus M and J are the only two parameters we need to fully classify black holes.

E. Cosmic Censorship Hypothesis

Rotating black holes with $a > \frac{r_0}{2}$ would have a singularity that in principle is accessible to us; it is not shielded by an event horizon, which seems wrong. The *cosmic censorship hypothesis* conjectures that this is impossible, but it remains unproven – it may very well be that these naked singularities exist in nature.

F. Black Hole Mechanics

1. *Schwarzschild Black Hole*

The surface gravity of a black hole, κ, is defined as the magnitude of the acceleration required for an object to be stationary at the Schwarzschild radius of a black hole, measured by the distant observer:

$$\sqrt{g_{\mu\nu}a^\mu_{(t)}a^\nu_{(t)}} = \frac{r_0}{2r^2}$$

hence

$$\kappa = \frac{1}{4GM}$$

The next constant we need to define is the area of the black hole, which is just

$$\mathcal{A} = 4\pi r_0^2$$

using these quantities we can formulate the first law of black hole mechanics:

> **First Law of Black Hole Mechanics (Schwarzschild)**
> Consider a small perturbation to the black hole, such that when it settles down again to a new stationary state it is again described as a Schwarzschild Black Hole. Then the change in mass M and area $\mathcal{A}$ obey
>
> $$\delta M = \frac{\kappa}{8\pi G}\delta\mathcal{A}$$

2. *Kerr Black Hole*

For a Kerr black hole things become more complicated. The surface gravity becomes

$$\kappa = \frac{r_+ - r_-}{2r_0 r_+}$$

and the area we can get by fixing r and t in the line element, and then taking the limit $r \to r_+$

$$\mathrm{d}s^2 = \frac{(r_+^2 + a^2)^2}{\Sigma}\sin^2\theta\mathrm{d}\phi^2 + \Sigma\mathrm{d}\theta^2$$

the Jacobian is the square root of determinant of the metric

$$\mathcal{A} = 2\pi(r_+^2 + a^2)\int_0^\pi \mathrm{d}\theta\sin\theta = 4\pi(r_+^2 + a^2)$$

which can be rewritten to

$$\mathcal{A} = 8\pi G^2 M^2\left(1 + \sqrt{1 - \frac{J^2}{G^2M^4}}\right)$$

From which the first law for Kerr black holes follows

> **First Law of Black Hole Mechanics (Kerr)**
> Consider a small perturbation to a black hole. When it again settles down to a new stationary Kerr black hole the quantities M, J, Ω_H, κ and $\mathcal{A}$ obey
>
> $$\mathrm{d}M = \frac{\kappa}{8\pi G}\delta\mathcal{A} + \Omega_H\delta J$$
>
> where
>
> $$\Omega_H \equiv \frac{a}{r_+r_0}$$

Second Law of Black Hole Mechanics

The energy

$$E = -mg_{\mu\nu}T^\mu\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}, \qquad T^\mu = \delta_t^\mu$$

is a conserved quantity far away from the black hole. However, in the ergoregion of a Kerr black hole, a part of this energy is spent on the forced rotation. Due to frame dragging and the fact that t is no longer a good time coordinate it is not E that is conserved, but rather

$$\mathcal{E} = -mg_{\mu\nu}\chi^\mu\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}, \qquad \chi^\mu = \delta_t^\mu + \Omega_H\delta_\phi^\mu$$

these two quantities are related through the angular momentum:

$$\mathcal{E} = E - \Omega_H L \qquad L \equiv mg_{\phi\nu}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}$$

The energy $\mathcal{E}$ should be positive just before it reaches the surface of the black hole, which using the conservation of energy gives us that

$$\delta M > \Omega_H\delta J$$

which implies that

> **Second Law of Black Hole Mechanics**
> The area $\mathcal{A}$ of the event horizon of a black hole cannot decrease as a function of time (measured asymptotically) under any process that can be described by general relativity.

IX. COSMOLOGY

A. Friedmann-Lemaître-Robertson-Walker Metric

Let us assume there exists a universal time, *cosmic standard time*, t, which can parametrise the evolution of the universe. Let us use spherical coordinates for the spatial part:

$$x^\mu = (t, r, \theta, \phi)$$

Now we make two assumptions about the geometry of the entire universe:

- Homogeneity: The geometry should look the same everywhere for a given t – invariant under spatial translations

- ISOTROPY: Given a point x^i, the universe should look the same in all directions; spherical symmetry about every point in the universe.

Naturally this does not hold on small scales; we require scales of at least order 100 million light years. These assumptions are what is referred to as the *cosmological principle*.

Due to the cosmological principle a line-element must be of the form

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + g_{ij}\mathrm{d}x^i\mathrm{d}x^j$$

We see that stationary particles follow geodesics, provided that $\Gamma^\mu_{tt} = 0$. This is clearly satisfied because all terms that appear in the Christoffel symbol are either zero or constant ($g_{t\mu} = -\delta_{t\mu}$). For fixed time the metric $g_{\mu\nu}|_{t=t_0}$ should describe homogeneous, isotropic space. Let us consider only the spatial part of the metric

$$\mathrm{d}\sigma^2 = \gamma_{ij}\mathrm{d}x^i\mathrm{d}x^j$$

Requiring that this is homogeneous and isotropic means that it should be *maximally symmetric*. For such a space the spatial part of the Riemann curvature tensor takes the following form

$${}^{(\gamma)}R_{ijk\ell} = \frac{k}{a^2}\left(\gamma_{ik}\gamma_{j\ell} - \gamma_{i\ell}\gamma_{jk}\right)$$

where $k \in \{-1, 0, 1\}$ is a dimensionless parameter that describes the type of space the universe is. a is a length parameter which is independent of x^i, but *can* be time-dependent. Furthermore we can show that the Ricci tensor simplifies to

$${}^{(\gamma)}R_{ij} = \frac{2k}{a^2}\gamma_{ij}$$

using this we can solve the differential equations provided by the Christoffel symbols and derive the *FLRW metric*:

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a(t)^2\left[\frac{\mathrm{d}r^2}{1-kr^2} + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right)\right]$$

Let us change coordinates such that the singularity at $r = \frac{1}{k}$ disappears: $\mathrm{d}\chi = \frac{\mathrm{d}r}{\sqrt{1-kr^2}}$ giving us

$$r(\chi) = \begin{cases} \sin\chi & \text{for} \quad k = 1 \\ \chi & \text{for} \quad k = 0 \\ \sinh\chi & \text{for} \quad k = -1 \end{cases}$$

If $k = 1$ we say that the universe is *closed*: the metric describes a three-sphere. If $k = 0$ the universe is *flat* and finally if $k = -1$ we say the universe is *open*.

B. Hubble's Law

Depending on whether $\dot{a} \equiv \frac{\mathrm{d}a}{\mathrm{d}t}$ is positive, negative or zero, the FLRW metric describes an expanding, contracting or static universe respectively.

Let our solar system be at the origin of the coordinate system. This would imply that every null curve that reaches Earth travels roughly on a radial null curve. Hence

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a(t)^2\mathrm{d}\chi^2$$

Integrating and using that $a(t)$ is approximately constant on time scales of the order of electromagnetic radiation periods:

$$\frac{T_f}{a(t_f)} = \frac{T_i}{a(t_i)}$$

where $T_f - T_i$ is the period of light. Defining the redshift parameter:

$$z \equiv \frac{\lambda_{\text{received}} - \lambda_{\text{emitted}}}{\lambda_{\text{emitted}}}$$

we can show that this can also be written as

$$z = \frac{a(t_f) - a(t_i)}{a(t_i)}$$

we see that if $z = 0$ the universe is static, if the universe expands $z > 0$ which implies that the light is *redshifted* and, finally, if $z < 0$ the light is *blueshifted*, corresponding to a contracting universe.

Consider a light ray emitted from a nearby galaxy, near enough that we can approximate the expansion of the universe as linear, then

$$a(t_i) \approx a(t_f) + \dot{a}(t_f)(t_i - t_f)$$

and so

$$z \approx \frac{\dot{a}(t_f)}{a(t_f)}(t_f - t_i)$$

in natural units $t_f - t_i = L$, the distance to the galaxy, thus there is a linear relation between the redshift parameter and the distance to nearby galaxies:

HUBBLE'S LAW
For nearby galaxies the redshift parameter, z, is proportional to the distance, L, to the galaxy to a good approximation:

$$z \approx H_0 L$$

where $H_0 \equiv \frac{\dot{a}(t_f)}{a(t_f)}$ is called the *Hubble constant*.

For nearby galaxies the time delay between emission and reception is so small (compared to cosmic time scales), that we can effectively say that the measured Hubble constant is the current value, and that it doesn't change in time.

The Hubble constant has been measured numerous times, however, with conflicting results. According to Planck 2015 $H_0 = (67 \pm 0.46)\mathrm{km\,s^{-1}\,Mpc^{-1}}$, while local distance-ladder measurements give values around $H_0 = (74 \pm 1.4)\mathrm{km\,s^{-1}\,Mpc^{-1}}$. This tension in empirical data is referred to as the *Hubble tension*.

C. Friedmann Equations

Einstein's equations for the description of the universe are rather ugly, because we can no longer assume that $T_{\mu\nu} = 0$; there is matter, energy, pressure, etc. everywhere in the universe, which means the right-hand side of Einstein's equations is non-zero. We've already discussed two types of fluids that can contribute to the energy-momentum tensor, which is of the form

$$T^{\mu\nu} = (\rho + p)u^\mu u^\nu + p g^{\mu\nu}$$

however, now we will add dark energy, and will motivate its existence shortly.

Dust:	$p = 0$
Photons:	$p = \frac{1}{3}\rho$
Dark energy:	$p = -\rho$

When we derived Einstein's field equations we included a term which I chose to refer to as an integration constant:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu} + \Lambda g_{\mu\nu}$$

the Λ term can be thought of as an additional form of energy/momentum/pressure, which is there even if $T_{\mu\nu} = 0$. This is dark energy, and it contributes to the expansion of the universe:

$$\rho = -p = \frac{\Lambda}{8\pi G}$$

We can now solve Einstein's field equations for the expanding universe in the presence of dark energy, which results in the following.

The Friedmann Equations
The scale factor $a(t)$ obeys the *Friedmann Equation*:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2}$$

Additionally $a(t)$ obeys

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p)$$

This is sometimes referred to as the second Friedmann equation.

However, we can also look at the conservation of the energy-momentum tensor:

$$0 = D_\mu T^{\mu}{}_{t} = -\left(\dot{\rho} + 3(\rho + p)\frac{\dot{a}}{a}\right)$$

Energy-Momentum Conservation
The energy-momentum tensor is conserved if the energy density and pressure obey

$$\dot{\rho} = -\frac{3\dot{a}}{a}(\rho + p)$$

This equation holds for each type of cosmological fluid individually, and therefore also holds for the total universe. This is because in this model the fluids do not interact with each other.

It is noteworthy that the (first) Friedmann Equation together with energy-momentum conservation leads to the second Friedmann Equation.

D. Evolution of the Scale Factor

Let us introduce the time-dependent Hubble parameter $H(t) = \frac{\dot{a}}{a}$ as well as the critical density:

$$\rho_{\text{crit}} = \frac{3H^2}{8\pi G}$$

and the density parameter:

$$\Omega = \frac{8\pi G}{3H^2}\rho$$

with these the Friedmann Equation can be written

$$\Omega - 1 = \frac{k}{H^2 a^2}$$

thus we can associate the different values of Ω to the different models of the universe

$\Omega < 1$	$k = -1$	Open Universe
$\Omega = 1$	$k = 0$	Flat Universe
$\Omega > 1$	$k = 1$	Closed Universe

1. *Composite Model*

We will now consider a model of the universe, which consists of four different species of cosmological fluids

- Radiation, $\rho_{\rm R}$. $w = \frac{1}{3}$
- Baryonic Matter, $\rho_{\rm B}$. $w = 0$
- Dark Matter, $\rho_{\rm DM}$. $w = 0$
- Dark energy, ρ_{Λ}. $w = -1$

In this model the total energy-density is

$$\rho = \rho_{\rm R} + \rho_{\rm B} + \rho_{\rm DM} + \rho_{\Lambda}$$

and

$$\Omega = \Omega_{\rm R} + \Omega_{\rm B} + \Omega_{\rm DM} + \Omega_{\Lambda}$$

According to Planck 2015 the current values are

$\Omega_R^{(0)}$	<0.001
$\Omega_B^{(0)}$	0.05
$\Omega_{DM}^{(0)}$	0.26
$\Omega_\Lambda^{(0)}$	0.69
Total	1.00

The total density of the universe is measured to be $\rho^{(0)} = 8.6 \times 10^{-26}$kg m^{-3}.

The energy-momentum conservation condition for each species can be written as

$$\frac{\dot\rho}{\rho} = -3(1+w)\frac{\dot a}{a}$$

which is solved if we let

$$\rho(t) \propto a^{-n}$$

where

$$n_{\rm R} = 4, \quad n_{\rm B} = n_{\rm DM} = 3, \quad n_{\Lambda} = 0$$

Note that $H^2 = \frac{8\pi G}{3}\sum_i \rho_i > 0$, which is consistent with an expanding universe when we choose the positive branch. Additionally

$$\dot H = -4\pi G \sum_i (1+w_i)\rho_i \leq 0$$

which implies that the Hubble parameter decreases as a function of time.

The universe is dominated more and more by dark energy, thus we expect that in the distant future we will have

$$a(t) = a_{\Lambda} e^{H_{\Lambda} t}$$

2. *Past of the Universe*

As we saw, $a(t)$ is related to the redshift parameter:

$$\frac{a(t_0)}{a(t)} = 1 + z$$

where t_0 is the time now and t is the time which we are looking back at. Thus

$$\rho_{\rm R} = \Omega_{\rm R}^{(0)}(1+z)^4, \quad \rho_{\rm M} = \Omega_{\rm M}^{(0)}(1+z)^3, \quad \rho_{\Lambda} = \Omega_{\Lambda}^{(0)}$$

where M now includes baryonic and dark matter. For a universe dominated by an energy density we have that $\rho \propto a^{-n}$, which implies that $a(t) \propto (t - t_\star)^{\frac{2}{n}}$. This implies that there was a beginning of the universe. However, this is not the case for Λ-dominated universes, as their beginning lies at $t = -\infty$.

The dark energy dominated phase of the universe (now) lies between

$$\Lambda\text{-dominated} \quad 0 < z < 0.4$$

Whereas the matter dominated phase was at

$$\text{Matter} - \text{dominated} \quad 0.4 < z < 3600$$

and finally

$$\text{Radiation} - \text{dominated} \quad 3600 < z$$

3. *Temperature of the Universe*

Currently the temperature of the Universe is $T_0 = 2.726$K. As we look back in time, this temperature will increase with z, proportionally to $(1+z)$, as the temperature is inversely proportional to the wavelength of the cosmic background radiation:

$$T(z) = T_0(1+z)$$

X. LINEARISED GRAVITY

Consider the limit where $g_{\mu\nu}$ differs slightly from the Minkowski metric:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \qquad |h_{\mu\nu}| \ll 1$$

plugging this into the Christoffel symbol and only keeping terms of linear order or less:

$$\Gamma^{\rho}_{\mu\nu} = \frac{1}{2}\left(\partial_\mu h_\nu{}^\rho + \partial_\nu h_\mu{}^\rho - \partial^\rho h_{\mu\nu}\right)$$

The geodesic equation for massive objects:

$$\frac{\mathrm{d}^2 x^\rho}{\mathrm{d}\tau^2} = -\left(\partial_\mu h_\nu{}^\rho - \frac{1}{2}\partial^\rho h_{\mu\nu}\right)\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}$$

The Ricci tensor becomes

$$R_{\mu\nu} = \frac{1}{2}\left(\partial_\mu\partial_\rho h_\nu{}^\rho + \partial_\nu\partial_\rho h_\mu{}^\rho - \Box^2 h_{\mu\nu} - \partial_\mu\partial_\nu h_\rho{}^\rho\right)$$

where $\Box^2 = \partial_\mu\partial^\mu$. This expression for the curvature tensor only includes terms of linear order in h. We can freely gauge transform our metric without it affecting the curvature tensor (though it might look different, we just transformed it using the rules for a $(0,2)$-tensor). In a suitable gauge the Ricci tensor just becomes

$$R_{\mu\nu} = -\frac{1}{2}\Box^2 h_{\mu\nu}$$

in this gauge we have that

$$\partial^\mu\left(h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h_\rho{}^\rho\right) = 0$$

(up to linear order in h).

Linearised Einstein Field Equations:
In the weak field limit of gravity Einstein's field equations can be written as

$$\Box^2 h_{\mu\nu} = -16\pi G\left(T_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\eta^{\rho\sigma}T_{\rho\sigma}\right)$$

up to linear order in $h_{\mu\nu}$. We impose the Lorenz gauge on $h_{\mu\nu}$, which implies that

$$\partial^\mu\left(h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h_\rho{}^\rho\right) = 0$$

A. Gravitational Waves

In vacuum $T_{\mu\nu} = 0$ which implies that the metric must satisfy the d'Alembert equation:

$$\Box^2 h_{\mu\nu} = 0$$

We know the form of the solution to this:

$$h_{\mu\nu} = A_{\mu\nu}\exp\left(ik_\rho x^\rho\right)$$

the Lorenz gauge implies that

$$k^\mu A_{\mu\nu} = \frac{1}{2}k_\nu\eta^{\rho\sigma}A_{\rho\sigma}$$

Additionally the d'Alembert equation tells us that

$$k_\mu k^\mu = 0$$

Consider a wave going in the x^3 direction, for it we must have

$$k^\mu = (\omega, 0, 0, \omega)$$

which in turn implies that

$$A_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & A_{11} & A_{12} & 0 \\ 0 & A_{12} & -A_{11} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

clearly there are two polarisations, both of which are orthogonal to the direction of propagation – just like electromagnetic radiation. Setting

$$h_{11}(t) = B_1\cos(\omega t + \chi_1), \quad h_{12}(t) = B_2\cos(\omega t + \chi_2)$$

we can calculate the proper distance between two test particles whose position at $t = 0$ are known to be

$$(x_A, y_A), \qquad (x_A + L_0\cos\theta, y_A + L_0\sin\theta)$$

Relative Motion of Test Particles
Consider two test particles that lie in the plane perpendicular to the propagation of the gravitational wave. Their positions in this plane are given by Equation X A. The proper distance between these particles is given by

$$\frac{L(t)}{L_0} = \left[1 + \frac{1}{2}h_{11}(t)\cos(2\theta) + \frac{1}{2}h_{12}(t)\sin(2\theta)\right]$$

where

$$h_{11}(t) = B_1\cos(\omega t + \chi_1)$$
$$h_{12}(t) = B_2\cos(\omega t + \chi_2)$$

Using interferometry one can measure this effect, however, only for gravitational waves whose wavelengths are much greater than the size of the device used. The issue is that we need very large devices in order to obtain the measurement accuracy required to make these measurements, therefore, the size and energy of gravitational events that can be measured is limited.

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