Groups

A group consists of a set and a composition rule, often denoted $(G, \cdot)$, where G is the set and $\cdot$ is the composition rule. In order for $(G, \cdot)$ to be a group it needs to fulfil the following axioms*:

1. Associativity:

$$(g_a \cdot g_b) \cdot g_c = g_a \cdot (g_b \cdot g_c), \quad \forall g_a, g_b, g_c \in G$$

2. The identity, I, must exist, for which:

$$I \cdot g = g \cdot I, \quad I \in G, \quad \forall g \in G$$

3. Every element of the group must have an inverse

$$g \cdot g^{-1} = g^{-1} \cdot g = I, \quad \forall g \in G$$

It can be shown that every element has a *unique* inverse.

(4.) Closure: If g is in the group and h is in the group then $g \cdot h$ is in the group.

Generally groups are *not* commutative, that is

$$g_a \cdot g_b \neq g_b \cdot g_a \quad \text{(generally)}$$

However, if this is the case we refer to the group as an **abelian group**. For example rotations in two dimensions form an abelian group under composition of rotations. However, rotations in three dimensions are not commutative – see Figure 2 in chapter I.1.

A subgroup, $H \subset G$, is a group for which[†]

$$h \in H \implies h \in G$$

Note that the axioms for groups put significant constraints on H. For example if G has N elements, and H has M elements, then $M|N$ (that means that N divided by M is an integer), this is **Lagrange's theorem**.

Cyclic subgroups

Given a finite group, G, let us pick an element, g, and repeatedly multiply it with itself. $g^2 \in G$ by closure, and hence $g^3 \in G$ etc. There must exist some positive integer, k, such that $g^k = I$, because the group is finite. We have just created a cyclic subgroup of G, which has order k. So a cyclic subgroup of G is any subgroup that can be written in the form

$$\left\{I, g, g^2, \cdots, g^{k-1}\right\}, \quad \text{for } 0 < k \leq |G|$$

where $|G|$ denotes the number of elements in G (the order of G).

Direct product of groups

Given two groups, $(G, \cdot)$ and $(H, \circ)$, then the direct product $(G \otimes H, \bullet)$ is also a group. The elements of this group are

$$(g, h) \quad \text{for } g \in G, \quad h \in H$$

and composition occurs elementwise:

$$(g_1, h_1) \bullet (g_2, h_2) = (g_1 \cdot g_2, h_1 \circ h_2)$$

The following is true:

$$|G \otimes H| = |G||H|$$

For example **Klein's Vierergruppe** is a direct product group $\mathbb{Z}_2 \otimes \mathbb{Z}_2$. $\mathbb{Z}_2$ has elements $\{0, 1\}$ hence

$$\mathbb{Z}_2 \otimes \mathbb{Z}_2 = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$$

Under *elementwise addition*, modulo 2.

A. Zee is confusing and says that

$$\mathbb{Z}_2 \otimes \mathbb{Z}_2 = \left\{ \begin{pmatrix} -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$$

under *elementwise multiplication.* However, these two groups are in fact isomorphic to each other, and can therefore be thought of as the same group. In fact they are two representations of the same group structure.

Multiplication tables

Given a group of order n we can define its multiplication table, whose i-j-th entry is the composition $g_i \cdot g_j$. By the group axioms every element in the group appears *once and only once* in every column and every row. Additionally, the first row and column are filled out trivially, because these are the elements that have been multiplied by the identity. Let's suppose our group has 4 elements, I, A, B and C:

	I	A	B	C
I	I	A	B	C
A	A	A^2	AB	AC
B	B	BA	B^2	BC
C	C	CA	CB	C^2

where the composition is implied. In the appendix I will fill out this table.

Abelian groups' multiplication tables are symmetric.

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* Remember $\forall$ looks like an upside down A: it means "for all". and $\exists$ is a backwards E: it means "there exists".

[†] This is an implication, not a bi-implication. Thus if $h \in H$ then $h \in G$ (but not every element in G is in H...)

Homomorphism and Isomorphism

A map $\Phi : G \to G'$ is a **homomorphism** if

$$\Phi(g_1 g_2) = \Phi(g_1)\Phi(g_2), \quad g_1, g_2 \in G$$

This mapping is an *isomorphism* if it additionally is bijective. For example $\mathbb{Z}_6 \cong \mathbb{Z}_2 \otimes \mathbb{Z}_3$, where "$\cong$"means "is isomorphic to". But A. Zee doesn't write $\cong$, instead he uses $=$, which I find very confusing.

Important fact about groups

Every finite group is isomorphic to a subgroup of S_n for some n. Think about the multiplication table, and that the "title-row is the unpermuted group (it's just the group in some order). The first row (where we multiplied by I) is thus just the trivial permutation, the second row is some permutation etc. Therefore each element in the group corresponds to some permutation of the group – thus we can define an isomorphism from every group to a subgroup of S_n.

Finite Groups

The **permutation group**, S_n, describes the permutation of n numbers (or letters or objects). There are different ways of denoting the elements in S_n. Let us look at S_5 specifically. An element in S_5 might be

$$g = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 1 & 5 & 2 & 3 \end{pmatrix}$$

where this means that $1 \rightsquigarrow 4$, $2 \rightsquigarrow 1$, $3 \rightsquigarrow 5$, $4 \rightsquigarrow 2$ and $5 \rightsquigarrow 3$. One can also use cycle notation. Our example from before would be written

$$g = (142)(35)$$

because 1 goes to 4 which goes to 2 which goes to 1 and 3 goes to 5 which goes to 3.

Any permutation in S_n can be written as a product of n_j j-cycles, provided $\sum_j j n_j = n$. Given a cycle structure characterised by integers n_j, then there are

$$\mathcal{N} = \frac{n!}{\prod_j j^{n_j} n_j!}$$

cycles with this structure.

A_n is the subgroup of S_n that contains all of S_n's *even* permutations. The dimensions are

	d
S_n	$n!$
A_n	$\frac{n!}{2}$

Rules for multiplying cycles:

It's always easiest just to write it out. For example what is (135)(24)(521)(34)? Well

$$\begin{aligned}(135)(24)(521)(34) &= \left(\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 2 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 1 & 4 & 3 & 2 \end{smallmatrix}\right) \\ &= \left(\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 1 & 4 & 3 & 2 \\ 1 & 3 & 2 & 5 & 4 \end{smallmatrix}\right) = (1)(23)(45) = (23)(45)\end{aligned}$$

However there are tricks that can be used so one doesn't need to write it all out.

1. If two cycles do not have any numbers in common, then they commute.
2. If two cycles have one number in common then they can be contracted, for example $(12)(23) = (123)$. Also $(12)(23)(34) = (12)(234) = (1234)$
3. $(12)(21) = I$. Because swapping two things twice is the same as not doing anything

A two-cycle (cycle with two numbers in it) is *odd* and a three-cycle is even. Generally even numbered cycles are odd and odd numbered cycles are even. Another way of saying this is that a permutation is even if it decomposes into an even number of 2-cycles (and odd otherwise).

Equivalence Classes

Two elements, g and g', in a group G are equivalent (and hence in the same equivalence class) if

$$g' = f^{-1} g f$$

for some $f \in G$. Equivalence is transitive, thus if g and g' are equivalent and g' and g'' are equivalent then g and g'' are too.

1. In an abelian group every element is only equivalent to itself
2. The identity is always in its own equivalence class
3. The inverse of all elements in a class also form a class

Dihedral group D_n

The dihedral group describes rotations together with reflections of an n-sided polygon. Let R be the rotation through $2\pi/n$ radians and r the reflection through a *median* (if n is odd the median passes through the centre of the polygon and one vertex, if n is even the median passes through two vertices, each on opposite sides). The group can be represented by

$$D_n = \{I, R, R^2, \cdots, R^{n-1}, r, Rr, R^2 r, \cdots, R^{n-1} r\}$$

note that

$$R^n = I, \quad r^2 = I$$

The quaternionic group $\mathcal{Q}$

Hamilton generalised the imaginary unit with his quaternions; they are defined as

$$\begin{aligned} i^2 = j^2 = k^2 &= -1 \\ ij = -ji &= k \\ jk = -kj &= i \\ ki = -ik &= j \end{aligned}$$

which I suppose could be written as

$$\alpha\beta = \varepsilon_{\alpha\beta\gamma}\gamma - \delta_{\alpha\beta}$$

This group is not abelian (it is anticommutative).

The invariant subgroup

Let H be a subgroup of G. H is an **invariant subgroup** if

$$ghg^{-1} \in H, \quad \forall h \in H, \forall g \in G$$

This is sometimes also written as

$$gHg^{-1} = H, \quad g \in G$$

hence the subgroup H is invariant under similarity transformations. **Derived subgroups**

Given a group, G, then the **derived subgroup** is generated by all elements of the form

$$\{a^{-1}b^{-1}ab : a, b \in G\}$$

So we look at all the products $a^{-1}b^{-1}ab$ and let a and b range through all elements in G.

Rotations

We can expand infinitesimal rotations up to the linear term, that is

$$R(\theta) \approx I + A$$

where A is an infinitesimal matrix of order θ. Rotations must be orthogonal

$$R^T R = I = (I + A^T)(I + A) = I + A^T + A = I$$

again neglecting higher order terms, hence $A^T = -A$. If A is antisymmetric, then it is proportional to

$$\mathcal{J} \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

hence

$$R(\theta) \approx I + \theta\mathcal{J} = \begin{pmatrix} 1 & \theta \\ -\theta & 1 \end{pmatrix}$$

The full matrix is given by infinitely many infinitesimal rotations:

$$R(\theta) = \lim_{N\to\infty} \left(R\left(\frac{\theta}{N}\right)\right)^N = e^{\theta\mathcal{J}} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

This approach can also be generalised to higher dimensions, however our matrix $\mathcal{J}$ is replaced by numerous matrices (in d dimensions there are $\frac{1}{2}d(d-1)$ of these). For example in three dimensions

$$\mathcal{J}_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad \mathcal{J}_y = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad \mathcal{J}_z = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

In which case

$$R(\theta) = e^{\theta_x\mathcal{J}_x + \theta_y\mathcal{J}_y + \theta_z\mathcal{J}_z}$$

But physicists often define $J_k = -i\mathcal{J}_k$, hence

$$R(\theta) = e^{i\sum_k \theta_k J_k}$$

The matrices J are called the **generators** of the rotation group (and the Lie algebra). For higher dimensions we must use two indices to indicate which rotation is meant. For example J_{13} would be rotating the 1-3-plane about itself. The ab-th entry in the matrix J_{mn} is given by

$$(J_{mn})^{ab} = -i\left(\delta^{ma}\delta^{nb} - \delta^{mb}\delta^{na}\right) = -i\varepsilon^{abc}\varepsilon^{mnc}$$

this holds in all dimensions. These are the generators of $SO(N)$, the *special orthogonal group*. We denote its Lie algebra as $so(N)$. "Special"refers to the fact that the matrices have det $= 1$ and "orthogonal"refers to the orthogonality of all the group's elements. N specifies that the group describes rotations in N dimensions.

Lie Algebra

Generally rotations do not commute (again see Figure 2 in chapter I.1.), hence we define the *commutator*:

$$[A, B] \equiv AB - BA$$

In the case of the generators

$$[J_a, J_b] = \sum_c i\varepsilon_{abc}J_c$$

where ε_{abc} is the totally antisymmetric tensor (Levi-Civita). ε_{abc} is the **structure constant** of the Lie Algebra. Generally, however,

$$[T_a, T_b] = \sum_c if_{abc}T_c = if_{abc}T_c$$

where T_i are the generators of the Lie Algebra. Due to the fact that $[\cdot\,,\cdot]$ is antisymmetric (by definition), f_{abc} is also antisymmetric. Note that I have used Einstein's summation convention; I will try to avoid this at all costs to make things easier to read.

Representation Theory & Character Tables

The notion of a group and an algebra is quite abstract – the elements in these *can* be vectors and matrices, but they can be anything. However, representation theory is all about representing these elements by matrices or tensors. Let $D(g)$ be the matrix that represents the group element g, then it must be true that

$$D(g_1 g_2) = D(g_1)D(g_2)$$

Note that this is the definition of a homomorphism! Usually we deal with isomorphisms, because it makes sense to allocate a unique matrix to every group element.

At this stage there is a bit of confusion. The group elements are represented by matrices (and later on tensors), and these matrices act on vectors (lower rank tensors later on). For example rotations in 2D can be represented by

$$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

and these matrices act upon (and rotate) 2-vectors. We say that this representation is *furnished by vectors.*

Now that we have defined the representation, we can define the **character**, χ, of each element, g:

$$\chi^{(r)}(g) = \mathrm{Tr} D^{(r)}(g), \quad g \in G$$

The character is dependent on which representation we chose, which is why we need to include the superscript (r), which just tells us which representation we are looking at.

An important characteristic of the trace is that it is invariant under cyclic permutation, hence

$$\mathrm{Tr}(ABC) = \mathrm{Tr}(CAB) = \mathrm{Tr}(BCA)$$

Therefore every element in an equivalence class has the character:

$$\mathrm{Tr}(f^{-1}gf) = \mathrm{Tr}(ff^{-1}g) = \mathrm{Tr}(g)$$

Similarity transformations of any representation does not change the character:

$$D' = S^{-1}DS \;\; \rightsquigarrow \;\; \mathrm{Tr}\left(D'\right) = \mathrm{Tr}\left(S^{-1}DS\right) = \mathrm{Tr}\left(D\right)$$

Reducible and irreducible representations

Let's look at $SO(3)$. The trivial 1-dimensional representation $(D^{(1)}(g))$ is a representation of all groups (it fulfils all the requirements). Another representation of $SO(3)$ is the 3-dimensional representation $(D^{(3)}(g))$, which we know well (3×3 special orthogonal matrices). We can take the direct sum of these representations, for example

$$D(g) = D^{(1)}(g) \oplus D^{(1)}(g) \oplus D^{(3)}(g) \oplus D^{(3)}(g)$$
$$= \left(\begin{array}{c|c|c|c} D^{(1)}(g) & 0 & 0 & 0 \\ \hline 0 & D^{(1)}(g) & 0 & 0 \\ \hline 0 & 0 & D^{(3)}(g) & 0 \\ \hline 0 & 0 & 0 & D^{(3)}(g) \end{array}\right)$$

this is an 8-dimensional representation, yet it is not particularly interesting, because it decomposes into two 1-dimensional representations and two 3-dimensional representations. Therefore we say that this representation is **reducible**. We focus primarily on **irreducible** representations, as we can build all reducible representations using these.

Unitarity Theorem: All finite groups have unitary representations, that is representations for which

$$D(g)^{\dagger} D(g) = I, \quad \forall g \in G$$

In general this is *not* true for infinite groups. However if it does hold for an infinite group, we refer to the group as **compact**.

Schur's Lemma: If $D(g)$ is an irreducible representation of a finite group, G, and if there is some matrix, A, such that $AD(g) = D(g)A$ for all g, then $A = \lambda I$ for some constant λ.

That is, if $D(g)$ is an irreducible representation of a group, then *the only matrix that commutes with* $D(g)$ is (a constant times) the identity matrix.

The Great Orthogonality Theorem: Given a d-dimensional irreducible representation $D(g)$ of a finite group G, we have

$$\sum_g D^{\dagger}(g)^i_j D(g)^k_l = \frac{N(G)}{d} \delta^i_l \delta^k_j$$

Additionally for two representations r and s we have that

$$\sum_g D^{(r)\dagger}(g)^i_j D^{(s)}(g)^k_l = \frac{N(G)}{d_r} \delta^{rs} \delta^i_l \delta^k_j$$

Let $i = j$ and $k = l$ and sum over all values of i and k, then we get:

$$\sum_{g,i,k} D^{(r)\dagger}(g)^i_i D^{(s)}(g)^k_k = \sum_{i,k} \frac{N(G)}{d_r} \delta^{rs} \delta^{ik} \delta^{ki}$$

The left-hand side is the product of the traces of the two matrices and hence the product of the characters in each representation, and the right-hand side gives us d_r times what we had before (because i and k are summed over d_r different values), thus

$$\sum_c n_c(\chi^{(r)}(c))^*\chi^{(s)}(c) = N(G)\delta^{rs}$$

the n_c appears because we change our summation from going over each group element, to going over each class. The class, c, has n_c elements.

Now we are ready to introduce the **character table**. Begin by writing out each equivalence class, each in its own row of the table. Next add a column for each irreducible representation. Note that

$$N(R) = N(C)$$

The number of irreducible representations is equal to the number of equivalence classes. Hence the table has as many rows as it has columns – it is square. Now we fill in the elements: the element in row c and column r is equal to $\chi^{(r)}(c)$. For example, for A_4:

A_4 n_c	c	1	$1'$	$1''$	3
1	I	1	1	1	3
3	$(12)(34)$	1	1	1	-1
4	(123)	1	ω	ω^*	0
4	(132)	1	ω^*	ω	0

We can find ω by using the different rules we just discussed. The result is

$$\omega = e^{i\frac{2\pi}{3}}$$

In the appendix I fill out the character table for Q.

Using these we can also **test for reducibility:** Given a representation $D(g)$ its characters will be given by

$$\chi(c) = \sum_r n_r\chi^{(r)}(c)$$

For r ranging over all irreducible representations (This is naturally true – if $D(g)$ is irreducible then n_r is zero for all representations, except when it is one. If $D(g)$ is reducible it must decompose into irreducible representations, in which case this is most definitely true). It can be shown that

$$\sum_c n_c\chi^*(c)\chi(c) = N(G)\sum_r(n_r)^2$$

hence by taking the trace of our representation (and summing over all classes) we can establish which irreducible representations appear how many times in the direct sum!

Regular representation

Let $N(G)$ be the number of elements of the group (i.e. the order of the group). As discussed previously, every group is isomorphic to a subgroup of $S_{N(G)}$, and $S_{N(G)}$ has a defining representation with dimension $N(G)$. Hence every group has a representation of order $N(G)$, which is known as the **regular representation**. Note that the regular representation is reducible (usually). For the regular representation $n_r = d_r$, hence the regular representation decomposes into n_r terms of the representation r. The regular representation is block diagonal.

Here are all the rules we have for the character tables:

1. Dimensions of the irreducible representations
$$\sum_r d_r^2 = N(G)$$
2. Column orthogonality
$$\sum_c n_c\left(\chi^{(r)}(c)\right)^*\chi^{(s)}(c) = N(G)\delta^{rs}$$
3. Row orthogonality
$$\sum_r\left(\chi^{(r)}(c)\right)^*\chi^{(r)}(c') = \frac{N(G)}{n_c}\delta^{cc'}$$
4. The character table is square
$$N(C) = N(R)$$

Additionally we have two rules for the decomposition of reducible representations

1. $\sum_c n_c\chi^*(c)\chi(c) = N(G)\sum_r n_r^2$
2. $\sum_c n_c\left(\chi^{(r)}(c)\right)^*\chi(c) = N(G)n_r$

Reality

Representations fall into three groups: **real**, **pseudoreal** and **complex**. Real and pseudoreal representations have real characters, whereas complex representations have genuinely complex characters. This discussion begins when we realise that if $D(g)$ is a representation, then so is $D^*(g)$. The characters of $D^*(g)$:

$$\chi^{(r^*)}(g) = \mathrm{Tr}\left(D(g)^*\right) = \mathrm{Tr}\left(D(g)\right)^* = \left(\chi^{(r)}(g)\right)^*$$

We can check whether a given representation is real, using the **reality checker**:

$$\sum_{g\in G}\chi^{(r)}(g^2) = \eta^{(r)}N(G), \quad \text{with } \eta^{(r)} = \begin{cases}1 & \text{if } \textit{real}\\ 0 & \text{if } \textit{complex}\\ -1 & \text{if } \textit{pseudoreal}\end{cases}$$

Note that if a representation is real, it does not necessarily mean that all entries in the original matrices are real, but rather that there exists a similarity transformation that makes the representation matrices real.

For the case of pseudoreality it is true that

$$D(g)^* = SD(g)S^{-1}$$

i.e. there exists a similarity transformation that takes you from $D(g)$ to $D(g)^*$.

Number of square roots

Starting with the reality checker note that if an element f has σ_f square roots then it will appear σ_f times in the sum, hence

$$\sum_f \sigma_f \chi^{(r)}(f) = \eta^{(r)} N(G)$$

From here we can show that

$$\sigma_f = \sum_r \eta^{(r)} \chi^{(r)}(f)$$

Remember the sum on the right only goes over the irreducible representations (of course, otherwise there would be infinitely many terms).

Rotation Groups and Tensors $SO(3)$

For $SO(3)$ we don't need to worry about the difference between upper and lower indices, therefore in this entire section we will only use upper indices.

The representation of $SO(3)$ that we are best acquainted with is the defining representation, i.e. 3×3 (special orthogonal) matrices. These matrices operate on (and rotate) vectors, that is the vector element V^i is transformed as follows:

$$V^i \ \rightsquigarrow\ V'^i = \sum_j R^{ij} V^j$$

where i and j take on three different values. We say that this defining representation is *furnished* by vectors. Similarly for $SO(N)$ the rotation matrix is $N \times N$ and the indices take on N different values. However, this is not the most general form. Here we introduce the **tensor**:

$$T^{ij} \ \rightsquigarrow\ T'^{ij} = \sum_{kl} R^{ik} R^{jl} T^{kl}$$

The representation that we get from studying this transformation is furnished by rank-2[‡] tensors. Note that in the case of $SO(3)$ the indices can take on three values, hence the representation is 9-dimensional.

But are these representations reducible? It turns out there are two types of tensors that transform independently from one another under these transformations:

1. Traceless symmetric tensors
2. Antisymmetric tensors

This can be seen by defining

$$A^{ij} = T^{ij} - T^{ji}$$

this transforms like

$$\begin{aligned} A^{ij} \ \rightsquigarrow\ A'^{ij} &= T'^{ij} - T'^{ji} \\ &= \sum_{kl} R^{ik} R^{jl} T^{kl} - \sum_{kl} R^{jk} R^{il} T^{kl} \\ &= \sum_{kl} R^{ik} R^{jl} (T^{kl} - T^{lk}) \\ &= \sum_{kl} R^{ik} R^{jl} A^{kl} \end{aligned}$$

A^{ij} is an antisymmetric tensor. Next consider

$$S^{ij} = T^{ij} + T^{ji}$$

This transforms like

$$\begin{aligned} S^{ij} \ \rightsquigarrow\ S'^{ij} &= T'^{ij} + T'^{ji} \\ &= \sum_{kl} R^{ik} R^{jl} T^{kl} + \sum_{kl} R^{jk} R^{il} T^{kl} \\ &= \sum_{kl} R^{ik} R^{jl} (T^{kl} + T^{lk}) \\ &= \sum_{kl} R^{ik} R^{jl} S^{kl} \end{aligned}$$

Note that if $i = j$ we get

$$\begin{aligned} \sum_i S^{ii} \ \rightsquigarrow\ \sum_i S'^{ii} &= \sum_{kl} R^{ik} R^{il} S^{kl} \\ &= \sum_{kl} \left(R^T\right)^{ki} R^{il} S^{kl} \\ &= \sum_{kl} \left(R^{-1}\right)^{ki} R^{il} S^{kl} \\ &= \sum_{kl} \delta^{kl} S^{kl} = \sum_k S^{kk} \end{aligned}$$

Hence the trace is invariant under these transformations. Thus a symmetric tensor transforms as two independent parts, a traceless symmetric tensor and a trace. Therefore we tend to work with symmetric traceless tensors.

[‡] rank-2 just means that the tensor has 2 indices.

Dimensions of rank-2 tensors: The first is easy: the trace of a rank-2 tensor is a number, hence 1-dimensional. The antisymmetric tensors have $N-1$ numbers in the first row, $N-2$ in the second all the way down until there are 0 numbers left.

$$\sum_{n=0}^{N-1} n = \frac{1}{2}N(N-1)$$

thus antisymmetric rank-2 tensors are $\frac{1}{2}N(N-1)$-dimensional. Symmetric rank-2 tensors have the same number *plus* the diagonal; there are N numbers on the diagonal

$$\frac{1}{2}N(N-1) + N = \frac{1}{2}N(N+1)$$

thus symmetric rank-2 tensors are $\frac{1}{2}N(N+1)$-dimensional.

Therefore our 9×9 matrix from before (the rank-2 tensor representation of $SO(3)$) decomposes into an antisymmetric part, a symmetric traceless part and a trace, hence there exists a similarity transformation such that

$$S^{-1}D(R)S = \left(\begin{array}{c|c|c} 3\times 3 \text{ block} & 0 & 0 \\ \hline 0 & 1\times 1 \text{ block} & 0 \\ \hline 0 & 0 & 5\times 5 \text{ block} \end{array}\right)$$

Rank-n tensors in $SO(N)$

To generalise what we did before for rank-2 tensors, we will look at

$$T^{i_1 i_2 \cdots i_n}$$

where each of the indices $i_1, i_2, \ldots, i_n$ can take on values between 1 and N. These tensors transform very similarly to rank-2 tensors:

$$\begin{aligned} T^{i_1 i_2 \cdots i_n} &\rightsquigarrow T'^{i_1 i_2 \cdots i_n} \\ &= \sum_{j_1, j_2, \ldots, j_n} R^{i_1 j_1} R^{i_2 j_2} \ldots R^{i_n j_n} T^{j_1 j_2 \cdots j_n} \end{aligned}$$

As before these split into symmetric traceless tensors, antisymmetric tensors as well as a trace. However, note that now the trace is a rank-$(n-2)$ tensor. This brings us to the **contraction of indices**, and **dual tensors**. Contraction of indices occurs by using the Kronecker delta:

$$\sum_{i_\alpha, i_\beta} \delta^{i_\alpha i_\beta} T^{i_1 \cdots i_n} = \sum_{i_\alpha, i_\beta} T^{i_1 \cdots i_\alpha \cdots i_\alpha \cdots i_n} \rightsquigarrow$$

$$\sum_{i_\alpha, i_\beta} T'^{i_1 \cdots i_\alpha \cdots i_\alpha \cdots i_n}$$

$$= \sum_{j_\alpha, j_\beta} R^{i_1 j_1} \ldots R^{i_\alpha j_\alpha} \ldots R^{i_\alpha j_\beta} \ldots R^{i_n j_n} T^{j_1 \cdots j_\alpha \cdots j_\alpha \cdots j_n}$$

$$= \sum_{j_\alpha} \underbrace{R^{i_1 j_1} \ldots R^{i_n j_n}}_{\text{does not include } R^{i_\alpha j_\alpha} \text{ nor } R^{i_\beta j_\beta}.} T^{j_1 \cdots j_\alpha \cdots j_\alpha \cdots j_n}$$

In the final equality I used, as before, that: $R^{i_\alpha j_\alpha} R^{i_\alpha j_\beta} = \delta^{j_\alpha j_\beta}$. Hence by renaming the indices $k_1, k_2, \ldots, k_{n-2}$ it becomes clear that the Kronecker delta has contracted two indices (i_α and i_β) such that we are now left with a rank-$(n-2)$ tensor.

We create the **dual tensor** by using the totally antisymmetric tensor on an antisymmetric tensor, $A^{i_1 i_2}$ (this is not the general case, but it's useful to start with this). Note that the totally antisymmetric tensor in $SO(N)$ will have N indices. Remember we have to sum over repeated indices, as usual thus

$$\sum_{i_1, i_2} \varepsilon^{i_1 i_2 \cdots i_N} A^{i_1 i_2} = B^{j_1 j_2 \cdots j_{N-2}}$$

thus $A^{i_1 i_2}$'s dual tensor is $B^{j_1 j_2 \cdots j_{N-2}}$. Generally if we begin with a rank-p tensor, using ε on it will result in a rank-$(N-p)$ tensor.

Dimensions of rank-n tensors, whose indices can take on N values

As usual we must differentiate between antisymmetric tensors, symmetric traceless tensors and traces.

Antisymmetric

An antisymmetric tensor with n indices that can take on N values has the following dimension:

$$d = \frac{N!}{n!(N-n)!} = \binom{N}{n}$$

Symmetric Traceless

A symmetric tensor with n indices that can take on N

values has the following dimension:

$$\begin{aligned} d &= \frac{(N+n-1)!}{n!(N-1)!} - \frac{(N+n-3)!}{(n-2)!(N-1)!} \\ &= \binom{N+n-1}{n} - \binom{N+n-3}{n-2} \end{aligned}$$

SO(3)

We often look at $SO(3)$ specifically. First note that for $SO(3)$ it's only important to look at symmetric tensors (we can look at the dual of antisymmetric tensors, which generally has fewer indices, for example an antisymmetric tensor with 2-indices can be written as a tensor with one index). Remember A. Zee's inductive approach to solving this problem. Thus let us look at a symmetric traceless tensor in $SO(3)$ with j indices:

$$\begin{aligned} d &= \frac{(j+2)!}{j!\,2!} - \frac{j!}{(j-2)!\,2!} \\ &= \frac{1}{2}\left((j+1)(j+2) - j(j-1)\right) = 2j+1 \end{aligned}$$

which is related to there being $2j+1$ states in a spin[§]-j system!

Adjoint representation

The **adjoint representation** is defined as

$$(T^a)^{bc} = -if^{abc}$$

That is, the matrix T^a's b-c-th element is given by a constant times the structure constant. For example, the matrices J from Equation are the adjoint representation of $SO(3)$.

It can be shown that the adjoint representation indeed is a representation by using Jacobi's relation, which states that

$$[[A,B],C] + [[B,C],A] + [[C,A],B] = 0$$

which, using the definition of our Lie Algebra can be written as

$$f^{abd}f^{dcg} + f^{bcd}f^{dag} + f^{cad}f^{dbg} = 0$$

Now plugging our definition of the adjoint representation we can show that this holds.

Ladder operators and $so(3)$

We would like to represent $so(3)$ with matrices; what this means is that we find matrices J_i such that

$$[J_a, J_b] = \sum_c i\varepsilon_{abc} J_c$$

We know from before that there exists a $(2j+1)$-dimensional representation of $SO(3)$, for j non-negative. Hence there must exist a $(2j+1)$-dimensional representation of the Lie Algebra. We introduce the ladder operators (raising and lowering operators), which will help us determine the entries in matrices J_i:

$$J_\pm \equiv J_x \pm iJ_y$$

note that

$$[J_z, J_\pm] = \pm J_\pm, \quad [J_+, J_-] = 2J_z$$

By using smart equations involving these commutation relations we can show that[¶]

$$J_\pm |m\rangle = c_{m\pm1} |m\pm1\rangle = \sqrt{(j+1\pm m)(j\mp m)}\, |m\pm1\rangle$$

Using these coefficients and inverting the equations in Equation we can obtain an expression for J_x and J_y. Note that $|m\rangle$ is in J_z's eigenbasis, hence

$$J_z |m\rangle = m |m\rangle$$

Hence it is diagonal with the values of m on the diagonal. Remember the equation from Quantum Mechanics

$$Q_{ij} = \langle i| \hat{Q} |j\rangle$$

Multiplying two representations together (still $SO(3)$)

In general multiplying two irreducible representations together results in a reducible representation. Let us begin with totally symmetric traceless tensors, $S^{i_1 i_2 \cdots i_j}$ and $T^{k_1 k_2 \cdots k'_j}$. Now we begin by creating a new tensor

$$P^{i_1\cdots i_j k_1 \cdots k_{j'}} = S^{i_1 i_2 \cdots i_j} T^{k_1 k_2 \cdots k_{j'}}$$

which only has *separate* symmetry properties for its first j indices and its final j' indices. We begin by symmetrising this and removing the trace, which gives us a rank-$(j+j')$ tensor. Now we use ε^{ikl} on P. Note that i and k are respectively an index from S and T (they can't *both* be from either S or T, because these are symmetric by construction, and using ε on a symmetric tensor gives you *zero*). Thus we trade off two indices for one, giving us a new tensor with $(j+j'-1)$ indices. We symmetrise and remove the trace, giving us a traceless symmetric rank-$(j+j'-1)$ tensor. This is repeated until one of the tensors S or T runs out of indices, thus

$$j \otimes j' = (j+j') \oplus (j+j'-1) \oplus \cdots \oplus |j-j'|$$

[§] Spin, angular momentum and probably other quantum numbers...

[¶] Note that these coefficients terminate for $m = \pm j$. That is $J_+ |j\rangle = 0$ and $J_- |-j\rangle = 0$, hence there are $2j+1$ states, as expected.

$|j - j'|$ gives us the number of indices that remains on the tensor that has most indices. We know the dimension of each term on the right side (for example $2(j+j')+1$), and by adding these dimensions we can determine the dimension of the coupled system (direct product system):

$$d = \sum_{k=|j-j'|}^{j+j'} (2k+1) = (2j+1)(2j'+1)$$

Let us look at two spin-$\frac{1}{2}$ systems, as we did in Quantum Mechanics. For these $j = j' = \frac{1}{2}$, hence[**]

$$\frac{1}{2} \otimes \frac{1}{2} = 1 \oplus 0, \qquad (j \text{ and } j')$$

which means that when you couple two spin-$\frac{1}{2}$ systems you get a spin-1 and a spin-0 system! Recall from QM when we looked at the spin-spin coupling of an electron and proton of hydrogen in its ground state we saw that the four states $\left(\frac{1}{2} \cdot 2 + 1\right)^2 = 4$ split into a triplet (a spin-1 system has 3 states) and a singlet (spin-0 system has one state)[††].

How do we name irreducible representation?

Instead of denoting our representation by j, we can represent it by its dimension. Hence Equation becomes

$$2 \otimes 2 = 3 \oplus 1, \qquad \text{(dimensions)}$$

Which is nice because $2 \cdot 2 = 3 + 1$. This works only when we are looking at the dimensions of the representations, not the values of j and j'.

But which states?– Clebsch-Gordan decomposition

So we have seen that $j \otimes j'$ decomposes into a direct sum of systems with j'' between $|j - j'|$ and $j + j'$, and hence each one of these systems has between $2|j-j'|+1$ and $2(j+j')+1$ states. But which states go where and what are the states actually?

Well, the systems with spin-j and spin-j' have the states

$$|j,\, m\rangle\,, \quad |j',\, m'\rangle$$

respectively. We are taking the tensor product of these two systems, so why not denote the coupled system's states with

$$|j,\, m\rangle \otimes |j',\, m'\rangle$$

Griffiths writes $|j, j', m, m'\rangle$, but it's the same thing – what we called the *uncoupled basis* in QM2. We are about to find the coupled basis. It can be shown that the product rule holds for the operators operating on tensor products of states, which means that, for example

$$\begin{aligned} &J_z(|j,m\rangle \otimes |j',m'\rangle) \\ &= (J_z\,|j,m\rangle) \otimes |j',m'\rangle + |j,m\rangle \otimes (J_z\,|j',m'\rangle) \\ &= (m+m')(|j,m\rangle \otimes |j',m'\rangle) \end{aligned}$$

So let us do something smart. We start with the state $|j,j\rangle \otimes |j',j'\rangle$, and use the lowering operator:

$$\begin{aligned} &J_-(|j,j\rangle \otimes |j',j'\rangle) \\ &= c_{j-1}\,|j,j-1\rangle \otimes |j',j'\rangle + c_{j'-1}\,|j,j\rangle \otimes |j',j'-1\rangle \end{aligned}$$

And now we take that state and use the lowering operator once again. This will give us $2(j+j')+1$ states – the states in the representation we called $(j+j')$. But this is not all the states in the system! We are expecting $(2j+1)(2j'+1)$. So let us find a state that is orthogonal to $J_-(|j,j\rangle \otimes |j',j'\rangle)$ and repeat the procedure again. We do this until we have all the states. For example, if $j = j' = \frac{1}{2}$ we get

$$\left|\frac{1}{2},\frac{1}{2}\right\rangle \otimes \left|\frac{1}{2},\frac{1}{2}\right\rangle$$

$$\frac{1}{\sqrt{2}}\left(\left|\frac{1}{2},-\frac{1}{2}\right\rangle \otimes \left|\frac{1}{2},\frac{1}{2}\right\rangle + \left|\frac{1}{2},\frac{1}{2}\right\rangle \otimes \left|\frac{1}{2},-\frac{1}{2}\right\rangle\right)$$

$$\left|\frac{1}{2},-\frac{1}{2}\right\rangle \otimes \left|\frac{1}{2},-\frac{1}{2}\right\rangle$$

$$\frac{1}{\sqrt{2}}\left(\left|\frac{1}{2},-\frac{1}{2}\right\rangle \otimes \left|\frac{1}{2},\frac{1}{2}\right\rangle - \left|\frac{1}{2},\frac{1}{2}\right\rangle \otimes \left|\frac{1}{2},-\frac{1}{2}\right\rangle\right)$$

Four states, just as expected. The first three states are the states in the spin-1 system and the last state is the spin-0 system's state.

Special Unitary Groups $SU(N)$

The special unitary groups $SU(N)$ consist of $N \times N$ matrices, U, that are unitary, i.e. for which it is true that

$$U^\dagger U = I, \quad \det(U) = 1$$

A few important facts about $SU(N)$:

1. Now it is important whether we write indices on top or at the bottom. For example $\phi^i = (\phi_i)^*$
2. When we use the repeated index summation rule this only is true if one of the indices is on top and the other is at the bottom, that is

$$\sum_i \phi_i \psi^i$$

[**] I have added a parenthesis that says j and j' because the numbers in the equation are the values of j and j' (and linear combinations). In a second we will write a similar equation where the numbers instead represent the dimension of the representation.

[††] Page 176-177 in Griffiths QM 3rd ed.

3. The Kronecker delta now has a lower and an upper index:

$$\delta_i^j$$

4. The totally antisymmetric tensor comes in two forms

$$\varepsilon^{i_1 i_2\cdots i_n}, \quad \varepsilon_{i_1 i_2\cdots i_n}$$

Now, vector transformations look like

$$\psi^i \rightsquigarrow \psi'^i = \sum_j U^i_j \psi^j$$

Similarly we can define transformations of covectors

$$\psi_i \rightsquigarrow \psi'_i = \sum_j (U^\dagger)^j_i \psi_j$$

Thus tensors can have upper and lower indices, and these indices should be thought of as separate entities. Generally tensors transform like

$$T^{i_1 i_2\cdots i_n}_{j_1 j_2\cdots j_{n'}} \rightsquigarrow T'^{i_1 i_2\cdots i_n}_{j_1 j_2\cdots j_{n'}}$$
$$= \sum_{l_1 l_2\cdots l_n} \sum_{k_1 k_2\cdots k_{n'}} U^{i_1}_{k_1}\cdots U^{i_n}_{k_n} T^{k_1 k_2\cdots k_n}_{l_1 l_2\cdots l_{n'}} (U^\dagger)^{l_1}_{j_1}\cdots (U^\dagger)^{l_{n'}}_{j_{n'}}$$

But nobody can read that, so let's take an example: The tensor T^{ij}_{klm} transforms like

$$T^{ij}_{klm} \rightsquigarrow$$
$$T'^{ij}_{klm} = \sum_{nopqr} U^i_n U^j_o T^{no}_{pqr} (U^\dagger)^p_k (U^\dagger)^q_l (U^\dagger)^r_m$$

The rules from before about contraction of indices and dual tensors are *similar* to how they were in $SO(3)$ however, slightly more complicated. Now we can:

1. Contract an upper with a lower index using δ^i_k. So that means out of a tensor with j upper indices and j' lower indices we make a tensor with $(j-1)$ upper indices and $(j'-1)$ lower indices. I'll write it out for a specific case:

$$\sum_{i,k} \delta^i_k F^{ij}_k = \sum_i F^{ij}_i = G^j$$

so $\sum_{i,k} \delta^i_k F^{ij}_k$ transforms like a tensor with one upper index

2. Move indices from the top to the bottom or vice versa, for example

$$\sum_{ij} \varepsilon_{ijpq} \varphi^{ij}_k = \psi_{pqk}$$

so $\sum_{ij} \varepsilon_{ijpq} \varphi^{ij}_k$ transforms like a tensor with three lower indices.

Generators of $SU(N)$

It can be shown that the generators of $SU(N)$ are $N \times N$, Hermitian, traceless matrices (or at least can be represented by these). In general there are $N^2 - 1$ of these (because they also need to be linearly independent of one another). For example the generators of the defining 2-dimensional representation of $SU(2)$ are the Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

And for the defining 3-dimensional representation of $SU(3)$ they are the Gell-Mann matrices

$$\lambda_1 = \begin{pmatrix} 0&1&0\\1&0&0\\0&0&0 \end{pmatrix}, \lambda_2 = \begin{pmatrix} 0&-i&0\\i&0&0\\0&0&0 \end{pmatrix}, \quad \lambda_3 = \begin{pmatrix} 1&0&0\\0&-1&0\\0&0&0 \end{pmatrix},$$
$$\lambda_4 = \begin{pmatrix} 0&0&1\\0&0&0\\1&0&0 \end{pmatrix}, \lambda_5 = \begin{pmatrix} 0&0&-i\\0&0&0\\i&0&0 \end{pmatrix}, \quad \lambda_6 = \begin{pmatrix} 0&0&0\\0&0&1\\0&1&0 \end{pmatrix},$$
$$\lambda_7 = \begin{pmatrix} 0&0&0\\0&0&-i\\0&i&0 \end{pmatrix}, \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1&0&0\\0&1&0\\0&0&-2 \end{pmatrix}$$

Adjoint representation of $SU(N)$

Just like with $SO(N)$ we define the adjoint representation as

$$(T^a)^{bc} = -if^{abc}$$

The b-c-th element of the generator T^a is equal to a constant times the structure constants' a-b-c-th entry.

The structure constants for the Pauli-matrix normalisation of $SU(2)$ are $2\varepsilon_{ijk}$, so

$$[\sigma_a, \sigma_b] = \sum_c 2i\varepsilon_{abc}\sigma_c$$

An interesting fact about $SU(2)$ is that it is locally isomorphic to $SO(3)$ and that it covers $SO(3)$ doubly. Hence we can represent rotations of vectors in 3 dimensions using $SU(2)$! Let $X = x\sigma_1 + y\sigma_2 + z\sigma_3$ then, for example

$$U^\dagger X U = X', \quad U = e^{i\boldsymbol{\varphi}\cdot\boldsymbol{\sigma}/2}$$

will rotate X by an angle equal to $|\varphi|$ about the axis of rotation $\hat{\boldsymbol{\varphi}}$. When using the generators for rotations, for example

$$\exp(-i\theta J_x) J_y \exp(i\theta J_x) = \cos\theta J_y + \sin\theta J_z$$

That is this expression rotates J_y about the x-axis by the angle θ.

Facts about $SU(2)$:

1. It can be described only using symmetric tensors with upper indices

2. Its representations have dimension $2j + 1$ for $(2j) \in \mathbb{Z}$

3. Its integer-spin representations are real, and its half-integer-spin representations are pseudoreal.

SU(3)

Isospin and hypercharge

Subatomic particles' charge is described using two quantum numbers, their isospin, I_3 and their hypercharge Y. Their charge is then

$$Q = I_3 + \frac{1}{2}Y$$

I_3 and Y are the quantum numbers that describe these particles in this $SU(3)$ description, which has two commuting operators.

Dimensions of $SU(3)$ tensors

In contrast to $SU(2)$ there can be both upper and lower indices in $SU(3)$ (ε has three indices, hence we cannot move everything up to the upper index). Therefore it is important that we distinguish between upper and lower indices.

Let us denote **symmetric traceless tensors with m upper indices and n lower indices by (m, n)**

We have previously shown that a symmetric tensor with n indices has dimension $\frac{1}{2}(n+1)(n+2)$. Additionally we have shown that the dimension of a product is the product of dimensions, hence if we have a symmetric tensor with m upper indices and n lower indices we would expect to get $\frac{1}{4}(m+1)(n+1)(m+2)(n+2)$, but we need to remove the trace, which is a tensor with $(m-1)$ upper indices and $(n-1)$ lower indices, hence

$$\begin{aligned}\mathcal{D}(m,n) =& \frac{1}{4}(m+1)(n+1)(m+2)(n+2) \\ &- \frac{1}{4}mn(m+1)(n+1) \\ =& \frac{1}{2}(m+1)(n+1)(m+n+2)\end{aligned}$$

Here is the table from A. Zee depicting the dimensions of some of the lowest irreducible representations

(m,n)	$\mathcal{D}$
$(1,0)$	3
$(1,1)$	8
$(2,0)$	6
$(2,1)$	15
$(2,2)$	27
$(3,0)$	10

Tabel I: *Dimension of a few irreducible representations of $SU(3)$. m is the number of upper indices and n is the number of lower indices of the symmetric traceless tensor.*

Multiplying irreducible representations of $SU(3)$

Let us multiply two irreducible representations (m, n) and (m', n'). How do these decompose? Let's begin with $(1,0)$ and $(0,1)$, that is φ^i and ψ_j. When we take the tensor product of these we get a tensor T^i_j, which we symmetrise and remove the trace and get $(1,1)$. And what is left is the trace, which is a scalar $(0,0)$, thus we conclude

$$(1,0) \otimes (0,1) = (1,1) \oplus (0,0), \quad (m,\ n,\ m' \text{ and } n')$$

As before we can write this as

$$3 \otimes 3^* = 8 \oplus 1, \quad \text{(dimensions)}$$

where we differentiate between the two 3-dimensional representations, as one has one upper index, and the other one lower index (they are conjugate to each other). A more challenging example is $(1,1)$ and $(1,1)$, which decomposes into

$$\begin{aligned}&(1,1) \otimes (1,1) \qquad\qquad (m,\ n,\ m' \text{ and } n') \\ &= (2,2) \oplus (3,0) \oplus (0,3) \oplus (1,1) \oplus (1,1) \oplus (0,0)\end{aligned}$$

and once again

$$8 \otimes 8 = 27 \oplus 10 \oplus 10^* \oplus 8 \oplus 8 \oplus 1, \quad \text{(dimensions)}$$

By repeatedly symmetrising and subtracting traces we can do this for arbitrary (m, n) and (m', n'). But first let us introduce new notation: $(m + m', n + n') = (m, n; m', n')$, so $(m, n; m', n')$ is a traceless symmetric tensor with $(m + m')$ upper indices and $(n + n')$ lower indices. Then

$$\begin{aligned}(m,n) \otimes (m',n') = &(m,n;m',n') \\ &\oplus (m-1,n;m',n'-1) \\ &\oplus (m,n-1;m'-1,n') \\ &\oplus (m-1,n-1;m'-1,n'-1) \\ &\oplus (m,n-2;m'-2,n') \\ &\oplus \cdots \|\end{aligned}$$

the $\cdots \|$ indicates that the process continues, but that it may terminate before $(0,0)$, (it could for example terminate with $(1,0)$).

Weights and Roots – $SU(3)$

The Gell-Mann matrices are the eight generators in the defining representation of $SU(3)$, which is important in particle physics. Here I have written them out again:

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$
$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \ \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \ \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$
$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \ \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

Now note:

1. λ_3 and λ_8 are diagonal – they commute
2. λ_1, λ_2 and λ_3 form a $SU(2)$ subalgebra

These generators satisfy the commutation relation that defines the Lie Algebra

$$[T^a, T^b] = \sum_c i f^{abc} T^c$$

for this normalisation of $SU(3)$ the structure constants are

$$\begin{aligned} f^{123} &= 1 \\ f^{147} = -f^{156} = f^{246} = f^{257} = f^{345} = -f^{367} &= \frac{1}{2} \\ f^{458} = f^{678} &= \frac{\sqrt{3}}{2} \end{aligned}$$

and all other values *that are not related to these through permutations of indices* are zero. The following discussion will be about the generators T^a in general (not specific to any representation). We see (from the structure constants) that $[T^3, T^8] = 0$, hence they can be diagonalised simultaneously. The remaining six generators are related to lowering and raising operators, as we saw for $SO(3)$:

$$\begin{aligned} I_\pm &= T_1 \pm iT_2 \\ U_\pm &= T_6 \pm iT_7 \\ V_\pm &= T_4 \pm iT_5 \\ I_3 &= T_3 \\ Y &= \frac{2}{\sqrt{3}} T_8 \end{aligned}$$

The odd normalisation of Y has historical reasons. Remember that T_1, T_2 and T_3 formed a $SU(2)$ subalgebra? Well that means that

$$[I_3, I_\pm] = \pm I_\pm, \quad [I_+, I_-] = 2I_3$$

Here are the remaining commutation relations

$$\begin{aligned} [I_3, U_\pm] &= \mp\frac{1}{2}U_\pm, & [I_3, V_\pm] &= \pm\frac{1}{2}V_\pm \\ [Y, U_\pm] &= \pm U_\pm, & [Y, V_\pm] &= \pm V_\pm \\ [Y, I_\pm] &= 0, & [I_+, V_-] &= -U_- \\ [I_+, U_+] &= V_+ & [U_+, V_-] &= I_- \\ [I_+, V_+] &= 0 & [I_+, U_-] &= 0 \end{aligned}$$

$$\begin{aligned} [U_+, U_-] &= \frac{3}{2}Y - I_3 \equiv U_3 \\ [V_+, V_-] &= \frac{3}{2}Y + I_3 \equiv V_3 \\ [U_+, V_+] &= 0 \end{aligned}$$

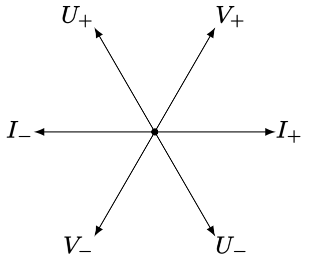
Root vectors

We are going to construct a lattice that represents all the states in the system. On the x-axis we will have i_3 (the eigenvalue of I_3) and on the y-axis we will have y (the hypercharge). Thus we call our states $|i_3, y\rangle$, and they are eigenstates of both I_3 and Y. The **root vectors** show in which direction the different raising and lowering operators take us:

$$I_\pm \leftrightsquigarrow \begin{pmatrix} \pm 1 \\ 0 \end{pmatrix}$$

$$U_\pm \leftrightsquigarrow \begin{pmatrix} \mp\frac{1}{2} \\ \pm 1 \end{pmatrix}$$

$$V_\pm \leftrightsquigarrow \begin{pmatrix} \pm\frac{1}{2} \\ \pm 1 \end{pmatrix}$$



Figur 1: *Root vectors for* $SU(3)$

Now that we know in which direction we can move, let us find out where we can start and stop, such that we will have found all the states. The states are represented by **weights** (vertices). But before we look at that let us define two terms:

1. Positive roots: roots whose first nonzero value is positive
2. Simple roots: positive roots that cannot be written as a sum of two positive roots *with positive coefficients.*

Our positive roots are V_+, I_+ and U_- and the simple roots are V_+ and U_-.

Next: weight diagrams are specific to which representation we are looking at! Remember we write our representation as (m, n) where m is the number of upper indices and n is the number of lower indices. Now it turns out that (m, n) also is the number of times we can use V_- and U_+ respectively on the **highest weight**[‡‡]. For

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[‡‡] The highest weight state is the state that is farthest along the line $i_3 = i_8$.

example in $(1,0)$, which is the defining 3-dimensional representation, we can use V_- once and U_+ no times. But how do we find the highest weight? You find it by using

$$\frac{\boldsymbol{\alpha}_i \cdot \boldsymbol{\mu}^{HW}}{\|\boldsymbol{\alpha}_i\|^2} = \frac{q_i}{2}$$

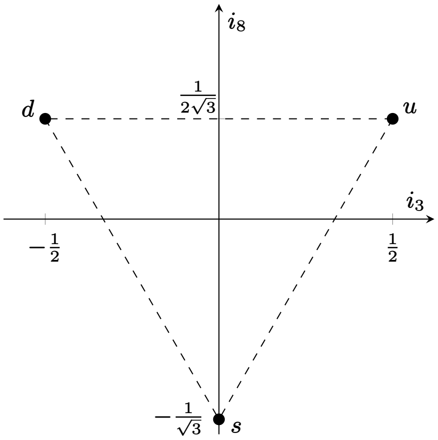
where $\boldsymbol{\alpha}_i$ is the i-th simple root and q_i is the i-th entry in our "vector"(m, n). Hence for $(1,0)$

$$\frac{1}{2}\left(\mu_1^{HW} + \sqrt{3}\mu_2^{HW}\right) = \frac{1}{2}$$
$$\frac{1}{2}\left(\mu_1^{HW} - \sqrt{3}\mu_2^{HW}\right) = 0$$

Hence $\boldsymbol{\mu}^{HW} = \left(\frac{1}{2}, \frac{1}{2\sqrt{3}}\right)$. Now we use two important facts

1. There is a $SU(2)$ subalgebra along the i_3-axis
2. We can use V_- and U_+ m and n times, respectively

Thus our weight diagram for $(1,0)$ is as follows



Figur 2: *Weight diagram for the* 3*-dimensional representation of* $SU(3)$*,* $(1,0)$*.* u *is the up quark,* d *the down quark and* s *the strange quark*

Here comes another important fact about weight diagrams. To get from a representation to its conjugate you mirror it along the i_3 axis. This can be seen by looking at the commutation relations

$$[D(T^a), D(T^b)] = \sum_c i f^{abc} D(T^c)$$

Here I am representing the T's by the matrices, D. Now let us take the complex conjugate[§§]

$$[D^*(T^a), D^*(T^b)] = -\sum_c i f^{abc} D^*(T^c)$$

which we can write as

$$[-D^*(T^a), -D^*(T^b)] = \sum_c i f^{abc} \left(-D^*(T^c)\right)$$

This changes the signs of *both* i_3 and i_8, however, note that the weight diagrams are always symmetric about the i_8-axis, therefore we effectively only mirror about the i_3 axis. You can find further weight diagrams in the appendix.

Weights and Roots – General Lie Algebras

First we must introduce new notation. Let H^i be the generators that commute, and let us call the remaining generators E^i. In the case of (the 3-dimensional representation of) $SU(3)$ we have that

$$H^1 = \frac{1}{\sqrt{2}}\text{diag}(1, -1, 0)$$
$$H^2 = \frac{1}{\sqrt{6}}\text{diag}(1, 1, -2)$$

That is to say the H's are diagonal matrices with the given values on the diagonal. The weights are found by reading off vertically, that is

$$w^1 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ \frac{1}{\sqrt{3}}\end{pmatrix},\ w^2 = \frac{1}{\sqrt{2}}\begin{pmatrix}-1\\ \frac{1}{\sqrt{3}}\end{pmatrix},\ w^3 = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\ -\frac{2}{\sqrt{3}}\end{pmatrix}$$

In our case here we only have three weights, so the six roots are easy to calculate (they are all the possible vectors that join the three points):

$$\alpha^1 = -\alpha^4 = w^1 - w^2 = \sqrt{2}\begin{pmatrix}1\\0\end{pmatrix}$$
$$\alpha^2 = -\alpha^5 = w^2 - w^3 = \frac{1}{\sqrt{2}}\begin{pmatrix}-1\\ \sqrt{3}\end{pmatrix}$$
$$\alpha^3 = -\alpha^6 = w^1 - w^3 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ \sqrt{3}\end{pmatrix}$$

Corresponding to $\sqrt{2}I_\pm$, $\sqrt{2}U_\pm$ and $\sqrt{2}V_\pm$ respectively. $SO(4)$ can be done similarly, and we end up with a

---

[§§] Note that $[A, B]^* = (AB - BA)^* = A^*B^* - B^*A^* = [A^*, B^*]$.

square weight diagram, that can be seen in the appendix

Generally for $SO(2\ell)$

The commuting generators are

$$\begin{aligned} H^1 &= \mathrm{diag}(1,-1,0,0,\cdots,0,0) \\ H^2 &= \mathrm{diag}(0,0,1,-1,0,0,\cdots,0,0) \\ &\vdots \\ H^\ell &= \mathrm{diag}(0,0,\cdots,0,0,1,-1) \end{aligned}$$

so the weights are

$$\begin{aligned} w^1 &= (1,0,\cdots,0) \\ w^2 &= (-1,0,\cdots,0) \\ w^3 &= (0,1,0,\cdots,0) \\ w^{2\ell-1} &= (0,\cdots,0,1) \\ w^{2\ell} &= (0,\cdots,0,-1) \end{aligned}$$

($\omega^i \in \mathbb{R}^\ell$), thus we can write the weights as $\pm e^i$ where e^i is the i-th unit vector in $\mathbb{R}^\ell$. Thus the roots are given by

$$\pm e^i \pm e^j \text{ (signs uncorrelated) } (i<j)$$

there are $4\ell(\ell-1)/2$ of these. We define the positive roots as $(e^i \pm e^j)$ again with $i<j$, which makes the simple roots:

$$e^i - e^{i+1} \quad (i=1,\ldots,\ell-1), \qquad e^{\ell-1}+e^\ell$$

And now $SO(2\ell+1)$ $SO(2\ell+1)$ is very similar to $SO(2\ell)$ but the vector representation has an additional* zero weight

$$w^{\ell+1} = (0,0,\ldots,0)$$

the roots for $SO(2\ell+1)$ are given by

$$\pm e^i \pm e^j \text{ (signs uncorrelated) } (i<j), \pm e^i$$

and hence the positive roots are $e^i \pm e^j$ and e^i, still with $i<j$, and the simple roots are

$$e^i - e^{i+1} \quad (i=1,\ldots,\ell-1), \qquad e^\ell$$

The roots of $SU(N)$

There are $\ell = N-1$ traceless matrices that commute with one another, these are

$$\begin{aligned} H^1 &= \frac{1}{\sqrt{2}}\mathrm{diag}\,(1,-1,0,\ldots,0) \\ H^2 &= \frac{1}{\sqrt{6}}\mathrm{diag}\,(1,1,-2,0,\ldots,0) \\ &\vdots \\ H^i &= \frac{1}{\sqrt{i(i+1)}}\mathrm{diag}\left(\underbrace{1,1,1,\ldots,1}_{i},-i,0,\ldots,0\right) \\ &\vdots \\ H^{\ell-1} &= \frac{1}{\sqrt{(\ell-1)\ell}}\mathrm{diag}\,(1,1,\ldots,1,1,1,-(\ell-1),0) \\ H^\ell &= \frac{1}{\sqrt{\ell(\ell+1)}}\mathrm{diag}\,(1,1,\ldots,1,1,1,1,-\ell) \end{aligned}$$

which makes the weights

$$w^1 = \sqrt{2}\left(\frac{1}{2},\frac{1}{2\sqrt{3}},\ldots,\frac{1}{\sqrt{2m(m+1)}},\ldots,\frac{1}{\sqrt{2\ell(\ell+1)}}\right)$$

$$w^2 = \sqrt{2}\left(\frac{-1}{2},\frac{1}{2\sqrt{3}},\ldots,\frac{1}{\sqrt{2m(m+1)}},\ldots,\frac{1}{\sqrt{2\ell(\ell+1)}}\right)$$

$$w^3 = \sqrt{2}\left(0,-\frac{1}{\sqrt{3}},\ldots,\frac{1}{\sqrt{2m(m+1)}},\ldots,\frac{1}{\sqrt{2\ell(\ell+1)}}\right)$$

$$\vdots$$

$$w^{m+1} = \sqrt{2}\left(0,0,\ldots,0,\frac{-m}{\sqrt{2m(m+1)}},\ldots,\frac{1}{\sqrt{2\ell(\ell+1)}}\right)$$

$$\vdots$$

$$w^{\ell+1} = \sqrt{2}(0,0,0,0,0\ldots,\frac{-\ell}{\sqrt{2\ell(\ell+1)}})$$

That's chaos, so let's look at $SU(4)$. $\ell = N-1$, thus we expect 3 commuting operators:

$$\begin{aligned} H^1 &= \frac{1}{\sqrt{2}}\mathrm{diag}\,(1,-1,0,0) \\ H^2 &= \frac{1}{\sqrt{6}}\mathrm{diag}\,(1,1,-2,0) \\ H^3 &= \frac{1}{2\sqrt{3}}\mathrm{diag}\,(1,1,1,-3) \end{aligned}$$

---

* And all the other matrices get a column and row of zeros added to the end.

and the weights are

$$\begin{aligned} w^1 &= \frac{1}{\sqrt{2}}\left(1, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{6}}\right) \\ w^2 &= \frac{1}{\sqrt{2}}\left(-1, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{6}}\right) \\ w^3 &= \frac{1}{\sqrt{2}}\left(0, -\frac{2}{\sqrt{3}}, \frac{1}{\sqrt{6}}\right) \\ w^4 &= \frac{1}{\sqrt{2}}\left(0, 0, -\sqrt{\frac{3}{2}}\right) \end{aligned}$$

There are six roots, of which 3 are simple:

$$\begin{aligned} \alpha^1 &= \sqrt{2}(1,0,0) \\ \alpha^2 &= \frac{1}{\sqrt{2}}(-1,\sqrt{3},0) \\ \alpha^3 &= \frac{1}{\sqrt{3}}(0,-\sqrt{2},2) \end{aligned}$$

Generally for $SU(\ell+1)$ the ℓ simple roots are

$$\alpha^i = e^i - e^{i+1}, \quad i \in \{1, 2, \ldots, \ell\}$$

Symplectic Algebras ($\boldsymbol{Sp(2\ell)}$)

The commuting generators are

$$H^i = u^i \otimes \sigma_3 = \left(\begin{array}{c|c} u^i & 0 \\ \hline 0 & -u^i \end{array}\right)$$

where $(u^i)_{mn} = \delta^{im}\delta^{mn}$, i.e u^i is an $\ell \times \ell$ matrix with a single entry: a one in the i-th column and i-th row. For example for $Sp(4)$:

$$\begin{aligned} H^1 &= \operatorname{diag}(1,0,-1,0) \\ H^2 &= \operatorname{diag}(0,1,0,-1) \end{aligned}$$

which makes that weights

$$\begin{aligned} w^1 &= (1,0) \\ w^2 &= (0,1) \\ w^3 &= (-1,0) \\ w^4 &= (0,-1) \end{aligned}$$

and the positive roots are

$$\begin{aligned} \alpha^1 &= 2e^1 \\ \alpha^2 &= e^1 + e^2 \\ \alpha^3 &= 2e^2 \\ \alpha^4 &= e^1 - e^2 \end{aligned}$$

To summarise:

	# of generators	roots	simple roots
$SU(\ell)$	$\ell^2 - 1$	$e^i - e^j$	$e^i - e^{i+1}$
$SO(2\ell+1)$	$\ell(2\ell+1)$	$\pm e^i \pm e^j, \pm e^i$	$e^i - e^{i+1}, e^\ell$
$Sp(2\ell)$	$\ell(2\ell+1)$	$\pm e^i \pm e^j, \pm 2e^i$	$e^i - e^{i+1}, 2e^\ell$
$SO(2\ell)$	$\ell(2\ell-1)$	$\pm e^i \pm e^j$	$e^i - e^{i+1}, e^{\ell-1} + e^\ell$

Tabel II

Classification of Lie Algebras

Remember a Lie Algebra with n generators is defined by

$$[T^a, T^b] = \sum_c i f^{ab}_c T^c$$

f^{ab}_c is antisymmetric in ab, but it does not make sense to swap for example a and c around. We introduce now the **Cartan-Killing metric**:

$$g^{ab} \equiv \operatorname{Tr}\left(T^a T^b\right) = -\sum_{c,d} f^{ac}_d f^{bd}_c$$

In physics we restrict ourselves to Lie Algebras for which $g^{ab\dagger}$ is invertible

$$\sum_b g^{ab} g_{bc} = \delta^a_c$$

Cartan Subalgebra

As before we denote the commuting generators by H^i and the remaining generators by E, i.e we have

$$[H^i, H^j] = 0$$

At this stage A. Zee starts using T^i for the matrices that represent H^i, but I will use H^i because that's far less confusing, but beware that H^i is the generator and T_i is the matrix representation thereof. We know that these are diagonal, let us call the a-th diagonal element $\beta^i(a)$

$$(H^i)^a_b = \left(\begin{array}{c|c|c|c} \beta^i(1) & 0 & 0 & 0 \\ \hline 0 & \beta^i(2) & 0 & 0 \\ \hline 0 & 0 & \beta^i(3) & 0 \\ \hline 0 & 0 & 0 & \beta^i(4) \end{array}\right) = \beta^i(a)\delta^a_b$$

---

† A. Zee says that if g^{ab} is real and symmetric we can relate it, by a similarity transformation, to δ^{ab}, so that the space in question is effectively Euclidean flat space.

This means that

$$[H^i, X^a] = \sum_b \beta^i(a)\delta^a_b X^b = \beta^i(a)X^a$$

X^a is one of the non-commuting matrices, which we henceforth will denote by E_β. Additionally we will define the root vector $\boldsymbol{\beta}(a)$:

$$\boldsymbol{\beta}(a) \equiv \left(\beta^1(a), \beta^2(a), \dots, \beta^\ell(a)\right)$$

Remember before, when we took the i-diagonal entry in the H matrices and called that the root, that is exactly what this definition says. We have that

$$[H^i, E_\beta] = \beta^i E_\beta \leftrightsquigarrow [H^i, E_\beta^\dagger] = -\beta^i E_\beta^\dagger$$

where you get the right part by taking the complex conjugate. Note that this means that[‡]

$$E_\beta^\dagger \equiv E_{-\beta}$$

A cool fact about roots is that if E_α and E_β are roots, then so is $[E_\alpha, E_\beta]$:

$$\begin{aligned} [H^i, [E_\alpha, E_\beta]] &= -[E_\alpha, [E_\beta, H^i]] - [E_\beta, [H^i, E_\alpha]] \\ &= [E_\alpha, \beta^i E_\beta] - [E_\beta, \alpha^i E_\alpha] \\ &= (\boldsymbol{\alpha} + \boldsymbol{\beta})^i [E_\alpha, E_\beta] \end{aligned}$$

Thus $[E_\alpha, E_\beta]$ is associated with the root $(\boldsymbol{\alpha} + \boldsymbol{\beta})$. However if $(\boldsymbol{\alpha} + \boldsymbol{\beta})$ is not a root, then $[E_\alpha, E_\beta] = 0$. Thus $2\boldsymbol{\alpha}$ is not a root (all operators commute with themselves). The following is true about the commutators

$$\begin{aligned} [H^i, H^j] &= 0 \\ [H^i, E_\alpha] &= \alpha^i E_\alpha \\ [E_\alpha, E_\beta] &= N_{\alpha,\beta} E_{\alpha+\beta} \\ [E_\alpha, E_{-\alpha}] &= \alpha_i H^i \end{aligned}$$

where $N_{\alpha,\beta}$ is some constant and $\sum_i \alpha_i g^{ij} = \alpha^j$. Let's suppose that we start at some β and then use E_α on it; this will move us in the direction of α, but not indefinitely, because the number of weights is finite, hence the ladder must terminate at some point. Let p be the number of times we can use E_α and q the number of times we can use $E_{-\alpha}$. This allows us to write an inductive relation for $N_{\alpha,\beta}$, which leads to

$$2\frac{\langle \boldsymbol{\alpha} | \boldsymbol{\beta} \rangle}{\langle \boldsymbol{\alpha} | \boldsymbol{\alpha} \rangle} = q - p = n \in \mathbb{Z}$$

similarly

$$2\frac{\langle \boldsymbol{\alpha} | \boldsymbol{\beta} \rangle}{\langle \boldsymbol{\beta} | \boldsymbol{\beta} \rangle} = q' - p' = m \in \mathbb{Z}$$

Hence

$$\cos^2 \theta_{\alpha\beta} = \frac{mn}{4} \qquad \Longrightarrow \qquad 0 \leq mn \leq 4$$

where $\theta_{\alpha\beta}$ is the angle between $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$. Additionally this implies that

$$\rho_{\alpha\beta} = \frac{\langle \boldsymbol{\alpha} | \boldsymbol{\alpha} \rangle}{\langle \boldsymbol{\beta} | \boldsymbol{\beta} \rangle} = \frac{m}{n} \in \mathbb{Q}$$

This leads to the **Cartan-Killing classification of Lie Algebras:** We can determine the angles between the roots, which puts all[§] Lie Algebras in one of four groups; this is summarised in the following table

m	n	$\frac{\langle \boldsymbol{\alpha} \vert \boldsymbol{\alpha} \rangle}{\langle \boldsymbol{\beta} \vert \boldsymbol{\beta} \rangle}$	$\cos^2 \theta_{\alpha\beta}$	$\theta_{\alpha\beta}$
1	1	1	$\frac{1}{4}$	60°
2	1	2	$\frac{1}{2}$	45°
3	1	3	$\frac{1}{4}$	30°
2	2	1	1	0°

Tabel III: *Summary of the geometry of root diagrams.*

Note that with this table we can show that:

> There are at most two different lengths of roots in a root diagram.

A very useful tool when determining root diagrams is **Weyl reflection:** Given two roots $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ we define

$$\boldsymbol{\beta}' = \boldsymbol{\beta} - 2\frac{\langle \boldsymbol{\alpha} | \boldsymbol{\beta} \rangle}{\langle \boldsymbol{\alpha} | \boldsymbol{\alpha} \rangle}\boldsymbol{\alpha} = \boldsymbol{\beta} + (p - q)\boldsymbol{\alpha}$$

if $\boldsymbol{\beta}$ is a root, then so is $\boldsymbol{\beta}'$, this helps us draw root diagrams much quicker, as it gives us additional roots, where we know the direction and the magnitude. This can be interpreted geometrically as: mirroring $\boldsymbol{\beta}$ through the hyperplane that is orthogonal $\boldsymbol{\alpha}$ gives you $\boldsymbol{\beta}$.

You can find all possible root diagrams for rank-2 Lie Algebras in the appendix.

Note that the angle between two *simple* roots has to be obtuse or right:

$$\theta_{\boldsymbol{\alpha}\boldsymbol{\beta}} \in \left[\frac{\pi}{2}, \pi\right]$$

[‡] This is remnant of the $\hat{a}$ and $\hat{a}^\dagger$ for the quantum mechanical harmonic oscillator, and that their action is the opposite of one another (the one is the creation and the other the annihilation operator)

[§] There are probably some exceptions...

Dynkin Diagrams

Every simple root is assigned either a filled circle, ●, or an open circle, ○. Each circle is connected with zero, one, two or three lines. Here is how we determine which:

1. If the simple root is one of the *short roots*, the circle is filled. If it is one of the *long roots* it is open.
2. Number of lines between circles:
   - *a.* Zero if $\theta_{\alpha\beta} = 90°$
   - *b.* One if $\theta_{\alpha\beta} = 120°$
   - *c.* Two if $\theta_{\alpha\beta} = 135°$
   - *d.* Three if $\theta_{\alpha\beta} = 150°$

The number of lines between two circles corresponding to the simple roots α and β respectively is given by

$$\mathcal{N}_L(\alpha, \beta) = \left(2\frac{\langle\boldsymbol{\alpha}|\boldsymbol{\beta}\rangle}{\langle\boldsymbol{\alpha}|\boldsymbol{\alpha}\rangle}\right)\left(2\frac{\langle\boldsymbol{\alpha}|\boldsymbol{\beta}\rangle}{\langle\boldsymbol{\beta}|\boldsymbol{\beta}\rangle}\right) = pp'$$

(Note that q and q' are zero because the roots are simple) A few Dynkin Diagrams are shown in the appendix.

Here are a few rules for Dynkin Diagrams:

1. Cutting: If you cut any line between two circles the resulting two diagrams must be legal Dynkin Diagrams
2. No circle can have more than three lines
3. No loops

Cartan Matrix:

We define the **Cartan Matrix** as

$$A_{ij} \equiv 2\frac{\langle\boldsymbol{\alpha}_i|\boldsymbol{\alpha}_j\rangle}{\langle\boldsymbol{\alpha}_i|\boldsymbol{\alpha}_i\rangle}$$

Things no one can remember (Appendix)

Integers modulo n: $\mathbb{Z}_n$

The integers form a finite group under addition modulo n. For example

$$\mathbb{Z}_4 = \{0, 1, 2, 3\}$$

because all numbers are either $0, 1, 2$ or 3 mod 4. For example

$$2 + 3 \equiv 1 \mod 4$$

$\mathbb{Z}_n$ is isomorphic to the n roots of unity under multiplication. For example $\mathbb{Z}_4$ is isomorphic to

$$\{1, i, -1, -i\}$$

under multiplication (note that these are all the fourth roots of 1)

Multiplication table

	I	A	B	C
I	I	A	B	C
A	A	A^2	AB	AC
B	B	BA	B^2	BC
C	C	CA	CB	C^2

There are two possible ways of filling this table out (there are two distinct groups of order 4 up to isomorphism). Let us first assume that $A^2 = I$[¶], then $\{I, A\}$ form a subgroup. Then $AB = C$ because the row with AB has A and I, and the column with AB has B, by similar argument $AC = B$. Also $AB = BA$ and $AC = CA$, due to the symmetry, which isn't always the case. Now we are left with a choice, either $B^2 = C^2 = I$ or $BC = CB = I$. These yield two distinct groups:

	I	A	B	C
I	I	A	B	C
A	A	I	C	B
B	B	C	I	A
C	C	B	A	I

Tabel IV: *Multiplication table for any group that is isomorphic to $\mathbb{Z}_2 \otimes \mathbb{Z}_2$*

	I	A	B	C
I	I	A	B	C
A	A	I	C	B
B	B	C	A	I
C	C	B	I	A

Tabel V: *The multiplication table for the cyclic group of order 4. B is the generator.*

Character table Q

First we need to determine the number of equivalence classes. Note that

$$(-i)^2 = (-j)^2 = (-k)^2 = -1$$

[¶] For groups of even order there is at least one element that squares to the identity.

So let us just calculate all the possible similarity transformations

	1	-1	i	j	k
1^{-1} 1	1	-1	i	j	k
(-1)$^{-1}$ -1	1	-1	i	j	k
i^{-1} i	1	-1	i	$-j$	$-k$
j^{-1} j	1	-1	$-i$	j	$-k$
k^{-1} k	1	-1	$-i$	$-j$	k

Tabel VI: *For example the row* j^{-1} j *takes* 1,i,j *and* k *and calculates* $j^{-1}1j$, $j^{-1}ij$, $j^{-1}jj$ *and* $j^{-1}kj$ *respectively*

Hence we see that

$$i \equiv -i, \quad j \equiv -j, \quad k \equiv -k$$

hence we have five equivalence classes, and hence five irreducible representations ($N(C) = N(R)$). Now we use that

$$\sum_r d_r^2 = N(G)$$

This says we have five numbers whose sum of squares must be equal to eight (there are eight elements in the group). There is only one solution to this

$$1 + 1 + 1 + 1 + 2^2 = 8$$

Thus we can already fill out a lot in the character table

$\mathcal{Q}$ n_c	c	1	$1'$	$1''$	$1'''$	2
1	1	1	1	1	1	2
1	-1	1	w	x	y	z
2	i	1	α	β	γ	ε
2	j	1	a	b	c	d
2	k	1	Λ	Γ	Ξ	Σ

Tabel VII: *Character table for* $\mathcal{Q}$*, incomplete.*

First off we use

$$\sum_c n_c \chi^{(r)*}(c)\chi^{(s)}(c) = N(G)\delta^{rs}$$

which, when we let $r = s$ tells us that all entries in the columns $1'$, $1''$ and $1'''$ either are 1 or -1. Also $z = \pm 2$. Now if we use the same equation but choose $r = 1$ and $s = 2$ we see that $z = -2$, so that the sum is zero. Now let us use

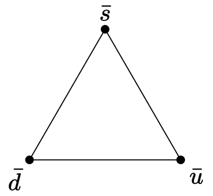
$$\sum_r \left(\chi^{(r)}(c)\right)^* \chi^{(r)}(c') = \frac{N(G)}{n_c}\delta^{cc'}$$

together with what we established previously tells us that $w = x = y = 1$. Now we know that the remaining entries have to have -1 twice in each row and column (due to the orthogonality), it doesn't matter in which order we do it, because we can freely swap the columns $1'$, $1''$ and $1'''$ around. Therefore the complete character table is

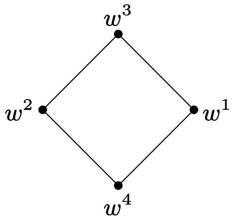
$\mathcal{Q}$ n_c	c	1	$1'$	$1''$	$1'''$	2
1	1	1	1	1	1	2
1	-1	1	1	1	1	-2
2	i	1	1	-1	-1	0
2	j	1	-1	1	-1	0
2	k	1	-1	-1	1	0

Tabel VIII: *Character table for* $\mathcal{Q}$*, complete.*

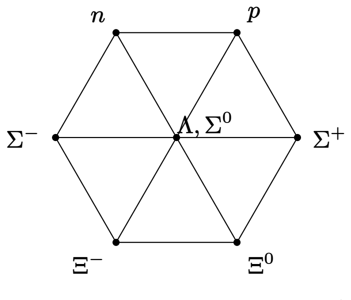
Further Weight Diagrams



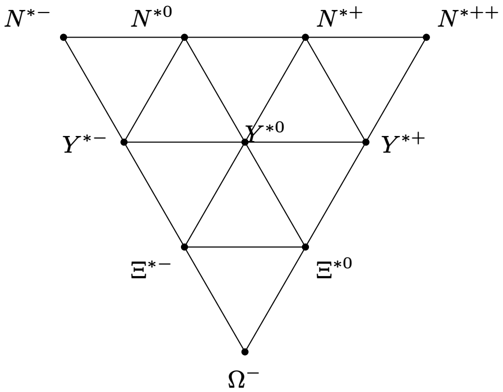
Figur 3: 3^**-dimensional representation of* $SU(3)$*,* $(0,1)$



Figur 4: 4*-dimensional representation of* $SO(4)$

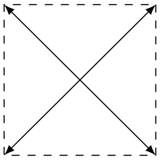


Figur 5: 8*-dimensional representation of* $SU(3)$*,* $(1,1)$*. Note that there are two distinct states in the middle,* Λ *and* Σ^0

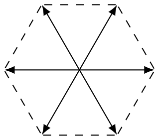


Figur 6: 10-*dimensional representation of* $SU(3)$, $(3,0)$

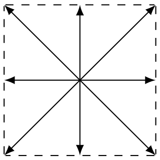
Root diagrams for rank-2 Lie Algebras



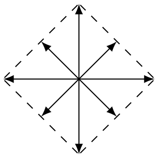
Figur 7: $SO(4) = D_4, \quad \theta_{\alpha\beta} = 90^\circ$



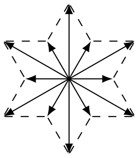
Figur 8: $SU(3) = A_2, \quad \theta_{\alpha\beta} = 60^\circ$



Figur 9: $SO(5) = B_2, \quad \theta_{\alpha\beta} = 45^\circ$

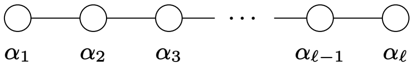


Figur 10: $Sp(4) = C_2, \quad \theta_{\alpha\beta} = 45^\circ$

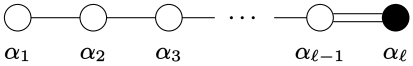


Figur 11: $G_2, \quad \theta_{\alpha\beta} = 30^\circ$

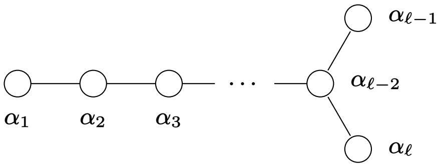
A Few Dynkin Diagrams



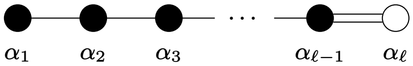
Figur 12: $SU(\ell+1)$



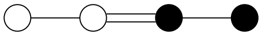
Figur 13: $SO(2\ell+1)$



Figur 14: $SO(2\ell)$. *If* $\ell = 2$ *then* $\alpha_{\ell-2}$ *does not exist, hence the two circles are not connected.*



Figur 15: $Sp(2\ell)$



Figur 16: F_4

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